Anti-de Sitter spacetime

This worksheet demonstrates a few capabilities of SageManifolds (version 1.0, as included in SageMath 7.5) in computations regarding anti-de Sitter spacetime.

Click here to download the worksheet file (ipynb format). To run it, you must start SageMath within the Jupyter notebook, via the command `sage -n jupyter`

NB: a version of SageMath at least equal to 7.5 is required to run this worksheet:

In [1]:
```
version()
```

Out[1]:
```
'SageMath version 7.5.1, Release Date: 2017-01-15'
```

First we set up the notebook to display mathematical objects using LaTeX rendering:

In [2]:
```
%display latex
```

We also define a viewer for 3D plots (use `'threejs'` or `'jmol'` for interactive 3D graphics):

In [3]:
```
viewer3D = 'tachyon' # must be 'threejs', 'jmol', 'tachyon' or None (de
fault)
```

Spacetime manifold

We declare the anti-de Sitter spacetime as a 4-dimensional differentiable manifold:

In [4]:
```
M = Manifold(4, 'M', r'\mathcal{M}')
print(M) ; M
```

```
4-dimensional differentiable manifold M
```

Out[4]: $\mathcal{M}$

We consider hyperbolic coordinates $(\tau, \rho, \theta, \phi)$ on $\mathcal{M}$. Allowing for the standard coordinate singularities at $\rho = 0$, $\theta = 0$ or $\theta = \pi$, these coordinates cover the entire spacetime manifold (which is topologically $\mathbb{R}^4$). If we restrict ourselves to *regular* coordinates (i.e. to considering only mathematically well defined charts), the hyperbolic coordinates cover only an open part of $\mathcal{M}$, which we call $\mathcal{M}_0$, on which ρ spans the open interval $(0, +\infty)$, θ the open interval $(0, \pi)$ and ϕ the open interval $(0, 2\pi)$. Therefore, we declare:

In [5]:
```
M0 = M.open_subset('M_0', r'\mathcal{M}_0' )
X_hyp.<ta,rh,th,ph> = M0.chart(r'ta:\tau rh:(0,+oo):\rho th:(0,pi):\the
ta ph:(0,2*pi):\phi')
print(X_hyp) ; X_hyp
```

```
Chart (M_0, (ta, rh, th, ph))
```

Out[5]: $(\mathcal{M}_0, (\tau, \rho, \theta, \phi))$

$\mathbb{R}^5$ as an ambient space

The AdS metric can be defined as that induced by the immersion of $\mathcal{M}$ in $\mathbb{R}^5$ equipped with a flat pseudo-Riemannian metric of signature $(-,-,+,+,+)$. We therefore introduce $\mathbb{R}^5$ as a 5-dimensional manifold covered by canonical coordinates:

In [6]:
```
R5 = Manifold(5, 'R5', r'\mathbb{R}^5')
X5.<U,V,X,Y,Z> = R5.chart()
print(X5) ; X5
```

```
Chart (R5, (U, V, X, Y, Z))
```

Out[6]: $\left(\mathbb{R}^5,(U,V,X,Y,Z)\right)$

The AdS immersion into $\mathbb{R}^5$ is defined as a differential map Φ from $\mathcal{M}$ to $\mathbb{R}^5$, by providing its expression in terms of $\mathcal{M}$'s default chart (which is X_hyp = $(\mathcal{M}_0,(\tau,\rho,\theta,\phi))$) and $\mathbb{R}^5$'s default chart (which is X5 = $(\mathbb{R}^5,(U,V,X,Y,Z))$):

In [7]:
```
var('b', domain='real')
assume(b>0)
Phi = M.diff_map(R5, [sin(b*ta)/b * cosh(rh),
                      cos(b*ta)/b * cosh(rh),
                      sinh(rh)/b *sin(th)*cos(ph),
                      sinh(rh)/b *sin(th)*sin(ph),
                      sinh(rh)/b *cos(th)],
                 name='Phi', latex_name=r'\Phi')
print(Phi) ; Phi.display()
```

```
Differentiable map Phi from the 4-dimensional differentiable manifold M
 to the 5-dimensional differentiable manifold R5
```

Out[7]:

$$\begin{array}{llcl} \Phi: & \mathcal{M} & \longrightarrow & \mathbb{R}^5 \\ \text{on } \mathcal{M}_0: & (\tau,\rho,\theta,\phi) & \longmapsto & (U,V,X,Y,Z) \\ & & & = \left(\frac{\cosh(\rho)\sin(b\tau)}{b}, \frac{\cos(b\tau)\cosh(\rho)}{b}, \frac{\cos(\phi)\sin(\theta)\sinh(\rho)}{b}, \frac{\sin(\phi)\sin(\theta}{b}\right. \end{array}$$

The constant b is a scale parameter. Considering AdS metric as a solution of vacuum Einstein equation with negative cosmological constant Λ, one has $b=\sqrt{-\Lambda/3}$.

Let us evaluate the image of a point via the map Φ:

In [8]:
```
p = M.point((ta, rh, th, ph), name='p') ; print(p)
```

```
Point p on the 4-dimensional differentiable manifold M
```

In [9]:
```
p.coord()
```

Out[9]: (τ,ρ,θ,ϕ)

In [10]:
```
q = Phi(p) ; print(q)
```

```
Point Phi(p) on the 5-dimensional differentiable manifold R5
```

In [11]:
```
q.coord()
```

Out[11]:

$$\left(\frac{\cosh(\rho)\sin(b\tau)}{b}, \frac{\cos(b\tau)\cosh(\rho)}{b}, \frac{\cos(\phi)\sin(\theta)\sinh(\rho)}{b}, \frac{\sin(\phi)\sin(\theta)\sinh(\rho)}{b}, \frac{\cos(\theta)\sinh(\rho)}{b}\right)$$

The image of $\mathcal{M}$ by the immersion Φ is a hyperboloid of one sheet, of equation $-U^2 - V^2 + X^2 + Y^2 + Z^2 = -b^{-2}$. Indeed:

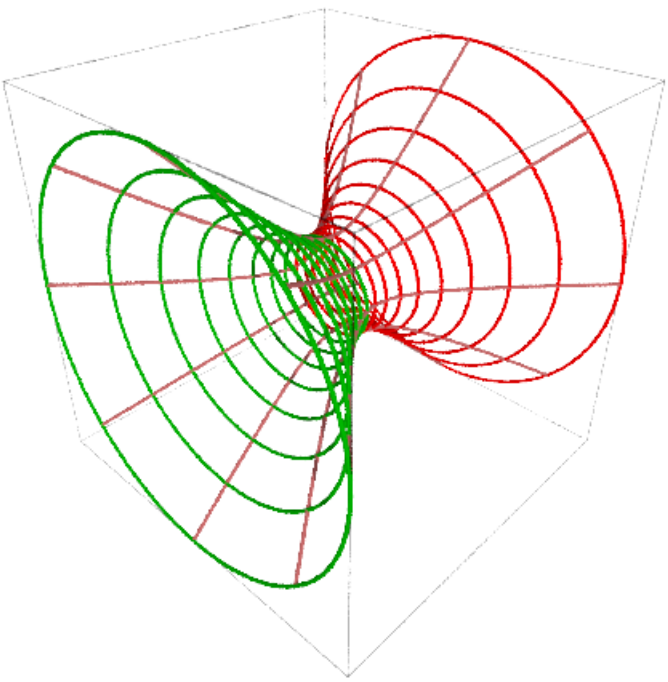
In [12]:
```
(Uq,Vq,Xq,Yq,Zq) = q.coord()
s = - Uq^2 - Vq^2 + Xq^2 + Yq^2 + Zq^2
s.simplify_full()
```

Out[12]:

$$-\frac{1}{b^2}$$

We may use the immersion Φ to draw the coordinate grid (τ, ρ) in terms of the coordinates (U, V, X) for $\theta = \pi/2$ and $\phi = 0$ (red) and $\theta = \pi/2$ and $\phi = \pi$ (green) (the brown lines are the lines $\tau = \mathrm{const}$):

In [13]:
```
graph1 = X_hyp.plot(X5, mapping=Phi, ambient_coords=(V,X,U), fixed_coor
ds={th:pi/2, ph:0},
                     ranges={ta:(0,2*pi), rh:(0,2)}, number_values=9,
                     color={ta:'red', rh:'brown'}, thickness=2, paramete
rs={b:1},
                     label_axes=False)
graph2 = X_hyp.plot(X5, mapping=Phi, ambient_coords=(V,X,U), fixed_coor
ds={th:pi/2, ph:pi},
                     ranges={ta:(0,2*pi), rh:(0,2)}, number_values=9,
                     color={ta:'green', rh:'brown'}, thickness=2, parame
ters={b:1},
                     label_axes=False)
show(graph1+graph2, aspect_ratio=1, viewer=viewer3D, axes_labels=['V','
X','U'])
```



Spacetime metric

First, we introduce on $\mathbb{R}^5$ the flat pseudo-Riemannian metric h of signature $(-,-,+,+,+)$:

In [14]:
```
h = R5.metric('h', signature=1)
h[0,0], h[1,1], h[2,2], h[3,3], h[4,4] = -1, -1, 1, 1, 1
h.display()
```

Out[14]: $h = -\mathrm{d}U \otimes \mathrm{d}U - \mathrm{d}V \otimes \mathrm{d}V + \mathrm{d}X \otimes \mathrm{d}X + \mathrm{d}Y \otimes \mathrm{d}Y + \mathrm{d}Z \otimes \mathrm{d}Z$

As mentionned above, the AdS metric g on $\mathcal{M}$ is that induced by h, i.e.g is the pullback of h by the map Φ:

In [15]:
```
g = M.lorentzian_metric('g')
g.set( Phi.pullback(h) )
```

The expression of g in terms of $\mathcal{M}$'s default frame is found to be

In [16]:
```
g.display()
```

Out[16]:

$$g = -\cosh(\rho)^2 \mathrm{d}\tau \otimes \mathrm{d}\tau + \frac{1}{b^2}\mathrm{d}\rho \otimes \mathrm{d}\rho + \frac{\sinh(\rho)^2}{b^2}\mathrm{d}\theta \otimes \mathrm{d}\theta + \frac{\sin(\theta)^2 \sinh(\rho)^2}{b^2}\mathrm{d}\phi \otimes \mathrm{d}\phi$$

In [17]:
```
g[:]
```

Out[17]:

$$\begin{pmatrix} -\cosh(\rho)^2 & 0 & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 & 0 \\ 0 & 0 & \frac{\sinh(\rho)^2}{b^2} & 0 \\ 0 & 0 & 0 & \frac{\sin(\theta)^2 \sinh(\rho)^2}{b^2} \end{pmatrix}$$

Curvature

The Riemann tensor of g is

In [18]:
```
Riem = g.riemann()
print(Riem)
Riem.display()
```

```
Tensor field Riem(g) of type (1,3) on the 4-dimensional differentiable
manifold M
```

Out[18]:

$$\begin{aligned}\mathrm{Riem}(g) = {} & -\frac{\partial}{\partial\tau} \otimes \mathrm{d}\rho \otimes \mathrm{d}\tau \otimes \mathrm{d}\rho + \frac{\partial}{\partial\tau} \otimes \mathrm{d}\rho \otimes \mathrm{d}\rho \otimes \mathrm{d}\tau - \sinh(\rho)^2 \frac{\partial}{\partial\tau} \otimes \mathrm{d}\theta \otimes \mathrm{d}\tau \otimes \mathrm{d}\theta + \sinh(\rho)^2 \frac{\partial}{\partial\tau} \otimes \mathrm{d}\theta \otimes \mathrm{d}\theta \otimes \mathrm{d}\tau \\ & - \sin(\theta)^2 \sinh(\rho)^2 \frac{\partial}{\partial\tau} \otimes \mathrm{d}\phi \otimes \mathrm{d}\tau \otimes \mathrm{d}\phi + \sin(\theta)^2 \sinh(\rho)^2 \frac{\partial}{\partial\tau} \otimes \mathrm{d}\phi \otimes \mathrm{d}\phi \otimes \mathrm{d}\tau - b^2 \cosh(\rho)^2 \frac{\partial}{\partial\rho} \otimes \mathrm{d}\tau \otimes \mathrm{d}\tau \otimes \mathrm{d}\rho \\ & + b^2 \cosh(\rho)^2 \frac{\partial}{\partial\rho} \otimes \mathrm{d}\tau \otimes \mathrm{d}\rho \otimes \mathrm{d}\tau - \sinh(\rho)^2 \frac{\partial}{\partial\rho} \otimes \mathrm{d}\theta \otimes \mathrm{d}\rho \otimes \mathrm{d}\theta + \sinh(\rho)^2 \frac{\partial}{\partial\rho} \otimes \mathrm{d}\theta \otimes \mathrm{d}\theta \otimes \mathrm{d}\rho \\ & - \sin(\theta)^2 \sinh(\rho)^2 \frac{\partial}{\partial\rho} \otimes \mathrm{d}\phi \otimes \mathrm{d}\rho \otimes \mathrm{d}\phi + \sin(\theta)^2 \sinh(\rho)^2 \frac{\partial}{\partial\rho} \otimes \mathrm{d}\phi \otimes \mathrm{d}\phi \otimes \mathrm{d}\rho - b^2 \cosh(\rho)^2 \frac{\partial}{\partial\theta} \otimes \mathrm{d}\tau \otimes \mathrm{d}\tau \otimes \mathrm{d}\theta \\ & + b^2 \cosh(\rho)^2 \frac{\partial}{\partial\theta} \otimes \mathrm{d}\tau \otimes \mathrm{d}\theta \otimes \mathrm{d}\tau + \frac{\partial}{\partial\theta} \otimes \mathrm{d}\rho \otimes \mathrm{d}\rho \otimes \mathrm{d}\theta - \frac{\partial}{\partial\theta} \otimes \mathrm{d}\rho \otimes \mathrm{d}\theta \otimes \mathrm{d}\rho - \sin(\theta)^2 \sinh(\rho)^2 \frac{\partial}{\partial\theta} \otimes \mathrm{d}\phi \otimes \mathrm{d}\theta \otimes \mathrm{d}\phi \\ & + \sin(\theta)^2 \sinh(\rho)^2 \frac{\partial}{\partial\theta} \otimes \mathrm{d}\phi \otimes \mathrm{d}\phi \otimes \mathrm{d}\theta - b^2 \cosh(\rho)^2 \frac{\partial}{\partial\phi} \otimes \mathrm{d}\tau \otimes \mathrm{d}\tau \otimes \mathrm{d}\phi + b^2 \cosh(\rho)^2 \frac{\partial}{\partial\phi} \otimes \mathrm{d}\tau \otimes \mathrm{d}\phi \otimes \mathrm{d}\tau \\ & + \frac{\partial}{\partial\phi} \otimes \mathrm{d}\rho \otimes \mathrm{d}\rho \otimes \mathrm{d}\phi - \frac{\partial}{\partial\phi} \otimes \mathrm{d}\rho \otimes \mathrm{d}\phi \otimes \mathrm{d}\rho + \sinh(\rho)^2 \frac{\partial}{\partial\phi} \otimes \mathrm{d}\theta \otimes \mathrm{d}\theta \otimes \mathrm{d}\phi - \sinh(\rho)^2 \frac{\partial}{\partial\phi} \otimes \mathrm{d}\theta \otimes \mathrm{d}\phi \otimes \mathrm{d}\theta\end{aligned}$$

In [19]:
```
Riem.display_comp(only_nonredundant=True)
```

Out[19]:

$$\begin{array}{lcl} \mathrm{Riem}(g)^{\tau}{}_{\rho\,\tau\,\rho} & = & -1 \\ \mathrm{Riem}(g)^{\tau}{}_{\theta\,\tau\,\theta} & = & -\sinh(\rho)^2 \\ \mathrm{Riem}(g)^{\tau}{}_{\phi\,\tau\,\phi} & = & -\sin(\theta)^2\sinh(\rho)^2 \\ \mathrm{Riem}(g)^{\rho}{}_{\tau\,\tau\,\rho} & = & -b^2\cosh(\rho)^2 \\ \mathrm{Riem}(g)^{\rho}{}_{\theta\,\rho\,\theta} & = & -\sinh(\rho)^2 \\ \mathrm{Riem}(g)^{\rho}{}_{\phi\,\rho\,\phi} & = & -\sin(\theta)^2\sinh(\rho)^2 \\ \mathrm{Riem}(g)^{\theta}{}_{\tau\,\tau\,\theta} & = & -b^2\cosh(\rho)^2 \\ \mathrm{Riem}(g)^{\theta}{}_{\rho\,\rho\,\theta} & = & 1 \\ \mathrm{Riem}(g)^{\theta}{}_{\phi\,\theta\,\phi} & = & -\sin(\theta)^2\sinh(\rho)^2 \\ \mathrm{Riem}(g)^{\phi}{}_{\tau\,\tau\,\phi} & = & -b^2\cosh(\rho)^2 \\ \mathrm{Riem}(g)^{\phi}{}_{\rho\,\rho\,\phi} & = & 1 \\ \mathrm{Riem}(g)^{\phi}{}_{\theta\,\theta\,\phi} & = & \sinh(\rho)^2 \end{array}$$

The Ricci tensor:

In [20]:
```
Ric = g.ricci()
print(Ric)
Ric.display()
```

```
Field of symmetric bilinear forms Ric(g) on the 4-dimensional different
iable manifold M
```

Out[20]:

$$\mathrm{Ric}(g) = 3\,b^2\cosh(\rho)^2 \mathrm{d}\tau\otimes\mathrm{d}\tau - 3\mathrm{d}\rho\otimes\mathrm{d}\rho - 3\,\sinh(\rho)^2\mathrm{d}\theta\otimes\mathrm{d}\theta - 3\,\sin(\theta)^2\sinh(\rho)^2\mathrm{d}\phi\otimes\mathrm{d}\phi$$

In [21]:
```
Ric[:]
```

Out[21]:

$$\begin{pmatrix} 3\,b^2\cosh(\rho)^2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3\,\sinh(\rho)^2 & 0 \\ 0 & 0 & 0 & -3\,\sin(\theta)^2\sinh(\rho)^2 \end{pmatrix}$$

The Ricci scalar:

In [22]:
```
R = g.ricci_scalar()
print(R)
R.display()
```

```
Scalar field r(g) on the 4-dimensional differentiable manifold M
```

Out[22]:

$$\begin{array}{llcl} \mathrm{r}(g): & \mathcal{M} & \longrightarrow & \mathbb{R} \\ \text{on } \mathcal{M}_0: & (\tau,\rho,\theta,\phi) & \longmapsto & -12\,b^2 \end{array}$$

We recover the fact that AdS spacetime has a constant curvature. It is indeed a **maximally symmetric space**. In particular, the Riemann tensor is expressible as

$$R^i{}_{jlk} = \frac{R}{n(n-1)}\left(\delta^i{}_k g_{jl} - \delta^i{}_l g_{jk}\right),$$

where n is the dimension of $\mathcal{M}$: $n = 4$ in the present case. Let us check this formula here, under the form $R^i{}_{jlk} = -\frac{R}{6} g_{j[k}\delta^i{}_{l]}$:

In [23]:
```
delta = M.tangent_identity_field()
Riem == - (R/6)*(g*delta).antisymmetrize(2,3)  # 2,3 = last positions o
f the type-(1,3) tensor g*delta
```

Out[23]: True

We may also check that AdS metric is a solution of the vacuum **Einstein equation** with (negative) cosmological constant:

In [24]:
```
Lambda = -3*b^2
Ric - 1/2*R*g + Lambda*g == 0
```

Out[24]: True

Spherical coordinates

Let us introduce spherical coordinates (τ, r, θ, ϕ) on the AdS spacetime via the coordinate change

$$r = \frac{\sinh(\rho)}{b}$$

In [25]:
```
X_spher.<ta,r,th,ph> = M0.chart(r'ta:\tau r:(0,+oo) th:(0,pi):\theta ph
:(0,2*pi):\phi')
print(X_spher) ; X_spher
```

```
Chart (M_0, (ta, r, th, ph))
```

Out[25]: $(\mathcal{M}_0, (\tau, r, \theta, \phi))$

In [26]:
```
hyp_to_spher = X_hyp.transition_map(X_spher, [ta, sinh(rh)/b, th, ph])
hyp_to_spher.display()
```

Out[26]:

$$\begin{cases} \tau & = & \tau \\ r & = & \frac{\sinh(\rho)}{b} \\ \theta & = & \theta \\ \phi & = & \phi \end{cases}$$

In [27]:
```
hyp_to_spher.set_inverse(ta, asinh(b*r), th, ph)
spher_to_hyp = hyp_to_spher.inverse()
spher_to_hyp.display()
```

Out[27]:

$$\begin{cases} \tau & = & \tau \\ \rho & = & \operatorname{arcsinh}(br) \\ \theta & = & \theta \\ \phi & = & \phi \end{cases}$$

The expression of the metric tensor in the new coordinates is

In [28]:
```
g.display(X_spher.frame(), X_spher)
```

Out[28]:

$$g = \left(-b^2r^2 - 1\right) \mathrm{d}\tau \otimes \mathrm{d}\tau + \left(\frac{1}{b^2r^2+1}\right) \mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\theta \otimes \mathrm{d}\theta + r^2 \sin(\theta)^2 \mathrm{d}\phi \otimes \mathrm{d}\phi$$

Similarly, the expression of the Riemann tensor is

In [29]:
```
Riem.display_comp(X_spher.frame(), X_spher, only_nonredundant=True)
```

Out[29]:

$$\begin{array}{lcl}
\mathrm{Riem}(g)^{\tau}{}_{r\,\tau\,r} & = & -\frac{b^2}{b^2r^2+1} \\
\mathrm{Riem}(g)^{\tau}{}_{\theta\,\tau\,\theta} & = & -b^2r^2 \\
\mathrm{Riem}(g)^{\tau}{}_{\phi\,\tau\,\phi} & = & -b^2r^2\sin(\theta)^2 \\
\mathrm{Riem}(g)^{r}{}_{\tau\,\tau\,r} & = & -b^4r^2 - b^2 \\
\mathrm{Riem}(g)^{r}{}_{\theta\,r\,\theta} & = & -b^2r^2 \\
\mathrm{Riem}(g)^{r}{}_{\phi\,r\,\phi} & = & -b^2r^2\sin(\theta)^2 \\
\mathrm{Riem}(g)^{\theta}{}_{\tau\,\tau\,\theta} & = & -b^4r^2 - b^2 \\
\mathrm{Riem}(g)^{\theta}{}_{r\,r\,\theta} & = & \frac{b^2}{b^2r^2+1} \\
\mathrm{Riem}(g)^{\theta}{}_{\phi\,\theta\,\phi} & = & -b^2r^2\sin(\theta)^2 \\
\mathrm{Riem}(g)^{\phi}{}_{\tau\,\tau\,\phi} & = & -b^4r^2 - b^2 \\
\mathrm{Riem}(g)^{\phi}{}_{r\,r\,\phi} & = & \frac{b^2}{b^2r^2+1} \\
\mathrm{Riem}(g)^{\phi}{}_{\theta\,\theta\,\phi} & = & b^2r^2
\end{array}$$