Kerr spacetime

This worksheet demonstrates a few capabilities of SageManifolds (version 1.0, as included in SageMath 7.5) in computations regarding Kerr spacetime.

Click here to download the worksheet file (ipynb format). To run it, you must start SageMath within the Jupyter notebook, via the command `sage -n jupyter`

NB: a version of SageMath at least equal to 7.5 is required to run this worksheet:

In [1]:
```
version()
```

Out[1]:
```
'SageMath version 7.5.1, Release Date: 2017-01-15'
```

First we set up the notebook to display mathematical objects using LaTeX rendering:

In [2]:
```
%display latex
```

We also define a viewer for 3D plots (use `'threejs'` or `'jmol'` for interactive 3D graphics):

In [3]:
```
viewer3D = 'jmol' # must be 'threejs', jmol', 'tachyon' or None (defaul
t)
```

Since some computations are quite long, we ask for running them in parallel on 8 cores:

In [4]:
```
Parallelism().set(nproc=8)
```

Spacetime manifold

We declare the Kerr spacetime as a 4-dimensional diffentiable manifold:

In [5]:
```
M = Manifold(4, 'M', r'\mathcal{M}')
print(M)
```

```
4-dimensional differentiable manifold M
```

Let us use the standard **Boyer-Lindquist coordinates** on it, by first introducing the part $\mathcal{M}_0$ covered by these coordinates and then declaring a chart BL (for *Boyer-Lindquist*) on $\mathcal{M}_0$, via the method `chart()`, the argument of which is a string expressing the coordinates names, their ranges (the default is $(-\infty, +\infty)$) and their LaTeX symbols:

In [6]:
```
M0 = M.open_subset('M0', r'\mathcal{M}_0')
BL.<t,r,th,ph> = M0.chart(r't r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\p
hi')
print(BL) ; BL
```

```
Chart (M0, (t, r, th, ph))
```

Out[6]: $(\mathcal{M}_0, (t, r, \theta, \phi))$

In [7]:
```
BL[0], BL[1]
```

Out[7]: (t, r)

Metric tensor

The 2 parameters m and a of the Kerr spacetime are declared as symbolic variables:

In [8]:
```
var('m, a', domain='real')
```

Out[8]: (m, a)

Let us introduce the spacetime metric:

In [9]:
```
g = M.lorentzian_metric('g')
```

The metric is set by its components in the coordinate frame associated with Boyer-Lindquist coordinates, which is the current manifold's default frame:

In [10]:
```
rho2 = r^2 + (a*cos(th))^2
Delta = r^2 -2*m*r + a^2
g[0,0] = -(1-2*m*r/rho2)
g[0,3] = -2*a*m*r*sin(th)^2/rho2
g[1,1], g[2,2] = rho2/Delta, rho2
g[3,3] = (r^2+a^2+2*m*r*(a*sin(th))^2/rho2)*sin(th)^2
g.display()
```

Out[10]:

$$g = \left(\frac{2\,mr}{a^2\cos(\theta)^2 + r^2} - 1\right) \mathrm{d}t \otimes \mathrm{d}t + \left(-\frac{2\,amr\sin(\theta)^2}{a^2\cos(\theta)^2 + r^2}\right) \mathrm{d}t \otimes \mathrm{d}\phi + \left(\frac{a^2\cos(\theta)^2 + r^2}{a^2 - 2\,mr + r^2}\right) \mathrm{d}r \otimes \mathrm{d}r + \left(a^2\cos(\theta)^2 + r^2\right) \mathrm{d}\theta \otimes \mathrm{d}\theta + \left(-\frac{2\,amr\sin(\theta)^2}{a^2\cos(\theta)^2 + r^2}\right) \mathrm{d}\phi \otimes \mathrm{d}t + \left(\frac{2\,a^2mr\sin(\theta)^2}{a^2\cos(\theta)^2 + r^2} + a^2 + r^2\right)\sin(\theta)^2 \mathrm{d}\phi \otimes \mathrm{d}\phi$$

A matrix view of the components with respect to the manifold's default vector frame:

In [11]:
```
g[:]
```

Out[11]:

$$\begin{pmatrix} \frac{2\,mr}{a^2\cos(\theta)^2 + r^2} - 1 & 0 & 0 & -\frac{2\,amr\sin(\theta)^2}{a^2\cos(\theta)^2 + r^2} \\ 0 & \frac{a^2\cos(\theta)^2 + r^2}{a^2 - 2\,mr + r^2} & 0 & 0 \\ 0 & 0 & a^2\cos(\theta)^2 + r^2 & 0 \\ -\frac{2\,amr\sin(\theta)^2}{a^2\cos(\theta)^2 + r^2} & 0 & 0 & \left(\frac{2\,a^2mr\sin(\theta)^2}{a^2\cos(\theta)^2 + r^2} + a^2 + r^2\right)\sin(\theta)^2 \end{pmatrix}$$

The list of the non-vanishing components:

In [12]:
```
g.display_comp()
```

Out[12]:

$$
\begin{array}{lcl}
g_{\,t\,t} & = & \frac{2\,mr}{a^2\,\cos\left(\theta\right)^2+r^2} - 1 \\
g_{\,t\,\phi} & = & -\frac{2\,amr\,\sin\left(\theta\right)^2}{a^2\,\cos\left(\theta\right)^2+r^2} \\
g_{\,r\,r} & = & \frac{a^2\,\cos\left(\theta\right)^2+r^2}{a^2-2\,mr+r^2} \\
g_{\,\theta\,\theta} & = & a^2\,\cos\left(\theta\right)^2 + r^2 \\
g_{\,\phi\,t} & = & -\frac{2\,amr\,\sin\left(\theta\right)^2}{a^2\,\cos\left(\theta\right)^2+r^2} \\
g_{\,\phi\,\phi} & = & \left(\frac{2\,a^2mr\,\sin\left(\theta\right)^2}{a^2\,\cos\left(\theta\right)^2+r^2} + a^2 + r^2\right)\sin\left(\theta\right)^2
\end{array}
$$

Levi-Civita Connection

The Levi-Civita connection ∇ associated with g:

In [13]:
```
nabla = g.connection() ; print(nabla)
```

```
Levi-Civita connection nabla_g associated with the Lorentzian metric g
on the 4-dimensional differentiable manifold M
```

Let us verify that the covariant derivative of g with respect to ∇ vanishes identically:

In [14]:
```
nabla(g) == 0
```

Out[14]: True

Another view of the above property:

In [15]:
```
nabla(g).display()
```

Out[15]: $\nabla_g g = 0$

The nonzero Christoffel symbols (skipping those that can be deduced by symmetry of the last two indices):

In [16]:
```
g.christoffel_symbols_display()
```

Out[16]:

$$\Gamma^{t}{}_{t\,r} = \frac{a^2 m r^2 + m r^4 - \left(a^4 m + a^2 m r^2\right) \cos(\theta)^2}{a^2 r^4 - 2\, m r^5 + r^6 + \left(a^6 - 2\, a^4 m r + a^4 r^2\right) \cos(\theta)^4 + 2\, \left(a^4 r^2 - 2\, a^2 m r^3 + a^2 r^4\right) \cos(\theta)^2}$$

$$\Gamma^{t}{}_{t\,\theta} = -\frac{2\, a^2 m r \cos(\theta) \sin(\theta)}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4}$$

$$\Gamma^{t}{}_{r\,\phi} = -\frac{\left(a^3 m r^2 + 3\, a m r^4 - \left(a^5 m - a^3 m r^2\right) \cos(\theta)^2\right) \sin(\theta)^2}{a^2 r^4 - 2\, m r^5 + r^6 + \left(a^6 - 2\, a^4 m r + a^4 r^2\right) \cos(\theta)^4 + 2\, \left(a^4 r^2 - 2\, a^2 m r^3 + a^2 r^4\right) \cos(\theta)^2}$$

$$\Gamma^{t}{}_{\theta\,\phi} = -\frac{2\, \left(a^5 m r \cos(\theta) \sin(\theta)^5 - \left(a^5 m r + a^3 m r^3\right) \cos(\theta) \sin(\theta)^3\right)}{a^6 \cos(\theta)^6 + 3\, a^4 r^2 \cos(\theta)^4 + 3\, a^2 r^4 \cos(\theta)^2 + r^6}$$

$$\Gamma^{r}{}_{t\,t} = \frac{a^2 m r^2 - 2\, m^2 r^3 + m r^4 - \left(a^4 m - 2\, a^2 m^2 r + a^2 m r^2\right) \cos(\theta)^2}{a^6 \cos(\theta)^6 + 3\, a^4 r^2 \cos(\theta)^4 + 3\, a^2 r^4 \cos(\theta)^2 + r^6}$$

$$\Gamma^{r}{}_{t\,\phi} = -\frac{\left(a^3 m r^2 - 2\, a m^2 r^3 + a m r^4 - \left(a^5 m - 2\, a^3 m^2 r + a^3 m r^2\right) \cos(\theta)^2\right) \sin(\theta)^2}{a^6 \cos(\theta)^6 + 3\, a^4 r^2 \cos(\theta)^4 + 3\, a^2 r^4 \cos(\theta)^2 + r^6}$$

$$\Gamma^{r}{}_{r\,r} = \frac{a^2 r - m r^2 + \left(a^2 m - a^2 r\right) \cos(\theta)^2}{a^2 r^2 - 2\, m r^3 + r^4 + \left(a^4 - 2\, a^2 m r + a^2 r^2\right) \cos(\theta)^2}$$

$$\Gamma^{r}{}_{r\,\theta} = -\frac{a^2 \cos(\theta) \sin(\theta)}{a^2 \cos(\theta)^2 + r^2}$$

$$\Gamma^{r}{}_{\theta\,\theta} = -\frac{a^2 r - 2\, m r^2 + r^3}{a^2 \cos(\theta)^2 + r^2}$$

$$\Gamma^{r}{}_{\phi\,\phi} = \frac{\left(a^4 m r^2 - 2\, a^2 m^2 r^3 + a^2 m r^4 - \left(a^6 m - 2\, a^4 m^2 r + a^4 m r^2\right) \cos(\theta)^2\right) \sin(\theta)^4 - \left(a^2 r^5 - 2\, m r^6 + r^7 + \left(a^6 r - 2\, a^4 m r^2 + a^4 r^3\right) \cos(\theta)^4 + 2\, \left(a^4 r^3 - 2\, a^2 m r^4 + a^2 r^5\right) \cos(\theta)^2\right) \sin(\theta)^2}{a^6 \cos(\theta)^6 + 3\, a^4 r^2 \cos(\theta)^4 + 3\, a^2 r^4 \cos(\theta)^2 + r^6}$$

$$\Gamma^{\theta}{}_{t\,t} = -\frac{2\, a^2 m r \cos(\theta) \sin(\theta)}{a^6 \cos(\theta)^6 + 3\, a^4 r^2 \cos(\theta)^4 + 3\, a^2 r^4 \cos(\theta)^2 + r^6}$$

$$\Gamma^{\theta}{}_{t\,\phi} = \frac{2\, \left(a^3 m r + a m r^3\right) \cos(\theta) \sin(\theta)}{a^6 \cos(\theta)^6 + 3\, a^4 r^2 \cos(\theta)^4 + 3\, a^2 r^4 \cos(\theta)^2 + r^6}$$

$$\Gamma^{\theta}{}_{r\,r} = \frac{a^2 \cos(\theta) \sin(\theta)}{a^2 r^2 - 2\, m r^3 + r^4 + \left(a^4 - 2\, a^2 m r + a^2 r^2\right) \cos(\theta)^2}$$

$$\Gamma^{\theta}{}_{r\,\theta} = \frac{r}{a^2 \cos(\theta)^2 + r^2}$$

$$\Gamma^{\theta}{}_{\theta\,\theta} = -\frac{a^2 \cos(\theta) \sin(\theta)}{a^2 \cos(\theta)^2 + r^2}$$

$$\Gamma^{\theta}{}_{\phi\,\phi} = -\frac{\left(\left(a^6 - 2\, a^4 m r + a^4 r^2\right) \cos(\theta)^5 + 2\, \left(a^4 r^2 - 2\, a^2 m r^3 + a^2 r^4\right) \cos(\theta)^3 + \left(2\, a^4 m r + 4\, a^2 m r^3 + a^2 r^4 + r^6\right) \mathrm{cc}}{a^6 \cos(\theta)^6 + 3\, a^4 r^2 \cos(\theta)^4 + 3\, a^2 r^4 \cos(\theta)^2 + r^6}$$

$$\Gamma^{\phi}{}_{t\,r} = -\frac{a^3 m \cos(\theta)^2 - a m r^2}{a^2 r^4 - 2\, m r^5 + r^6 + \left(a^6 - 2\, a^4 m r + a^4 r^2\right) \cos(\theta)^4 + 2\, \left(a^4 r^2 - 2\, a^2 m r^3 + a^2 r^4\right) \cos(\theta)^2}$$

$$\Gamma^{\phi}{}_{t\,\theta} = -\frac{2\, a m r \cos(\theta)}{\left(a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4\right) \sin(\theta)}$$

$$\Gamma^{\phi}{}_{r\,\phi} = -\frac{a^2 m r^2 + 2\, m r^4 - r^5 + \left(a^4 m - a^4 r\right) \cos(\theta)^4 - \left(a^4 m - a^2 m r^2 + 2\, a^2 r^3\right) \cos(\theta)^2}{a^2 r^4 - 2\, m r^5 + r^6 + \left(a^6 - 2\, a^4 m r + a^4 r^2\right) \cos(\theta)^4 + 2\, \left(a^4 r^2 - 2\, a^2 m r^3 + a^2 r^4\right) \cos(\theta)^2}$$

$$\Gamma^{\phi}{}_{\theta\,\phi} = \frac{a^4 \cos(\theta) \sin(\theta)^4 - 2\, \left(a^4 - a^2 m r + a^2 r^2\right) \cos(\theta) \sin(\theta)^2 + \left(a^4 + 2\, a^2 r^2 + r^4\right) \cos(\theta)}{\left(a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4\right) \sin(\theta)}$$

Killing vectors

The default vector frame on the spacetime manifold is the coordinate basis associated with Boyer-Lindquist coordinates:

In [17]:
```
M.default_frame() is BL.frame()
```

Out[17]: True

In [18]:
```
BL.frame()
```

Out[18]:

$$\left(\mathcal{M}_0, \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\right)\right)$$

Let us consider the first vector field of this frame:

In [19]:
```
xi = BL.frame()[0] ; xi
```

Out[19]:

$$\frac{\partial}{\partial t}$$

In [20]:
```
print(xi)
```

```
Vector field d/dt on the Open subset M0 of the 4-dimensional differenti
able manifold M
```

The 1-form associated to it by metric duality is

In [21]:
```
xi_form = xi.down(g) ; xi_form.display()
```

Out[21]:

$$\left(-\frac{a^2\cos(\theta)^2 - 2\,mr + r^2}{a^2\cos(\theta)^2 + r^2}\right)\mathrm{d}t + \left(-\frac{2\,amr\sin(\theta)^2}{a^2\cos(\theta)^2 + r^2}\right)\mathrm{d}\phi$$

Its covariant derivative is

In [22]:
```
nab_xi = nabla(xi_form) ; print(nab_xi) ; nab_xi.display()
```

```
Tensor field of type (0,2) on the Open subset M0 of the 4-dimensional d
ifferentiable manifold M
```

Out[22]:

$$\begin{aligned}
&\left(\frac{a^2m\cos(\theta)^2 - mr^2}{a^4\cos(\theta)^4 + 2\,a^2r^2\cos(\theta)^2 + r^4}\right)\mathrm{d}t\otimes\mathrm{d}r \\
&+ \left(\frac{2\,a^2mr\cos(\theta)\sin(\theta)}{a^4\cos(\theta)^4 + 2\,a^2r^2\cos(\theta)^2 + r^4}\right)\mathrm{d}t\otimes\mathrm{d}\theta \\
&+ \left(-\frac{a^2m\cos(\theta)^2 - mr^2}{a^4\cos(\theta)^4 + 2\,a^2r^2\cos(\theta)^2 + r^4}\right)\mathrm{d}r\otimes\mathrm{d}t \\
&+ \left(\frac{\left(a^3m\cos(\theta)^2 - amr^2\right)\sin(\theta)^2}{a^4\cos(\theta)^4 + 2\,a^2r^2\cos(\theta)^2 + r^4}\right)\mathrm{d}r\otimes\mathrm{d}\phi \\
&+ \left(-\frac{2\,a^2mr\cos(\theta)\sin(\theta)}{a^4\cos(\theta)^4 + 2\,a^2r^2\cos(\theta)^2 + r^4}\right)\mathrm{d}\theta\otimes\mathrm{d}t \\
&+ \left(\frac{2\left(a^3mr + amr^3\right)\cos(\theta)\sin(\theta)}{a^4\cos(\theta)^4 + 2\,a^2r^2\cos(\theta)^2 + r^4}\right)\mathrm{d}\theta\otimes\mathrm{d}\phi \\
&+ \left(-\frac{\left(a^3m\cos(\theta)^2 - amr^2\right)\sin(\theta)^2}{a^4\cos(\theta)^4 + 2\,a^2r^2\cos(\theta)^2 + r^4}\right)\mathrm{d}\phi\otimes\mathrm{d}r \\
&+ \left(-\frac{2\left(a^3mr + amr^3\right)\cos(\theta)\sin(\theta)}{a^4\cos(\theta)^4 + 2\,a^2r^2\cos(\theta)^2 + r^4}\right)\mathrm{d}\phi\otimes\mathrm{d}\theta
\end{aligned}$$

Let us check that the Killing equation is satisfied:

In [23]:
```
nab_xi.symmetrize() == 0
```
Out[23]: True

Similarly, let us check that $\frac{\partial}{\partial \phi}$ is a Killing vector:

In [24]:
```
chi = BL.frame()[3] ; chi
```
Out[24]: $\frac{\partial}{\partial \phi}$

In [25]:
```
nabla(chi.down(g)).symmetrize() == 0
```
Out[25]: True

Curvature

The Ricci tensor associated with g:

In [26]:
```
Ric = g.ricci() ; print(Ric)
```
```
Field of symmetric bilinear forms Ric(g) on the 4-dimensional different
iable manifold M
```

Let us check that the Kerr metric is a solution of the vacuum Einstein equation:

In [27]:
```
Ric == 0
```
Out[27]: True

Another view of the above property:

In [28]:
```
Ric.display()
```
Out[28]: $\mathrm{Ric}\left(g\right) = 0$

The Riemann curvature tensor associated with g:

In [29]:
```
R = g.riemann() ; print(R)
```
```
Tensor field Riem(g) of type (1,3) on the 4-dimensional differentiable
manifold M
```

Contrary to the Ricci tensor, the Riemann tensor does not vanish; for instance, the component $R^0{}_{123}$ is

In [30]:
```
R[0,1,2,3]
```
Out[30]:

$$-\frac{\left(a^{7} m - 2\, a^{5} m^{2} r + a^{5} m r^{2}\right) \cos\left(\theta\right) \sin\left(\theta\right)^{5} + \left(a^{7} m + 2\, a^{5} m^{2} r + 6\, a^{5} m r^{2} - 6\, a^{3} m^{2} r^{3} + 5\, a^{3} m r^{4}\right) \cos\left(\theta\right) \sin\left(\theta\right)^{3} - 2\left(a^{7} m - a^{5} m r^{2} - 5\, a^{3} m r^{4} - 3\, a m r^{6}\right) \cos\left(\theta\right) \sin\left(\theta\right)}{a^{2} r^{6} - 2\, m r^{7} + r^{8} + \left(a^{8} - 2\, a^{6} m r + a^{6} r^{2}\right) \cos\left(\theta\right)^{6} + 3\left(a^{6} r^{2} - 2\, a^{4} m r^{3} + a^{4} r^{4}\right) \cos\left(\theta\right)^{4} + 3\left(a^{4} r^{4} - 2\, a^{2} m r^{5} + a^{2} r^{6}\right) \cos\left(\theta\right)^{2}}$$

Bianchi identity

Let us check the Bianchi identity $\nabla_p R^i{}_{jkl} + \nabla_k R^i{}_{jlp} + \nabla_l R^i{}_{jpk} = 0$:

In [31]:
```
DR = nabla(R) ; print(DR)  #long (takes a while)
```

```
Tensor field nabla_g(Riem(g)) of type (1,4) on the 4-dimensional differ
entiable manifold M
```

In [32]:
```
for i in M.irange():
    for j in M.irange():
        for k in M.irange():
            for l in M.irange():
                for p in M.irange():
                    print DR[i,j,k,l,p] + DR[i,j,l,p,k] + DR[i,j,p,k,l]
 ,
```

```
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
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0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
```

If the last sign in the Bianchi identity is changed to minus, the identity does no longer hold:

In [33]:
```
DR[0,1,2,3,1] + DR[0,1,3,1,2] + DR[0,1,1,2,3] # should be zero (Bianchi
 identity)
```

Out[33]: 0

In [34]:
```
DR[0,1,2,3,1] + DR[0,1,3,1,2] - DR[0,1,1,2,3] # note the change of the
second + to -
```

Out[34]:

$$\frac{24\left(\left(a^5mr + a^3mr^3\right)\cos(\theta)\sin(\theta)^3 - \left(a^5mr - amr^5\right)\cos(\theta)\sin(\theta)\right)}{a^2r^6 - 2\,mr^7 + r^8 + \left(a^8 - 2\,a^6mr + a^6r^2\right)\cos(\theta)^6 + 3\left(a^6r^2 - 2\,a^4mr^3 + a^4r^4\right)\cos(\theta)^4 + 3\left(a^4r^4 - 2\,a^2mr^5 + a^2r^6\right)\cos(\theta)^2}$$

Kretschmann scalar

The tensor $R^\flat$, of components $R_{abcd} = g_{am} R^m_{\ \ bcd}$:

In [35]:
```
dR = R.down(g) ; print(dR)
```

```
Tensor field of type (0,4) on the 4-dimensional differentiable manifold
 M
```

The tensor $R^\sharp$, of components $R^{abcd} = g^{bp} g^{cq} g^{dr} R^a_{\ \ pqr}$:

In [36]:
```
uR = R.up(g) ; print(uR)
```

```
Tensor field of type (4,0) on the 4-dimensional differentiable manifold
 M
```

The Kretschmann scalar $K := R^{abcd} R_{abcd}$:

In [37]:
```
Kr_scalar = uR['^{abcd}']*dR['_{abcd}']
Kr_scalar.display()
```

Out[37]:

$$\begin{array}{llcl} & \mathcal{M} & \longrightarrow & \mathbb{R} \\ \text{on } \mathcal{M}_0 : & (t, r, \theta, \phi) & \longmapsto & -\frac{48 \left(a^6 m^2 \cos(\theta)^6 - 15\, a^4 m^2 r^2 \cos(\theta)^4 + 15\, a^2 m^2 r^4 \cos}{a^{12} \cos(\theta)^{12} + 6\, a^{10} r^2 \cos(\theta)^{10} + 15\, a^8 r^4 \cos(\theta)^8 + 20\, a^6 r^6 \cos(\theta)^6 + 15\, a^4 r} \end{array}$$

A variant of this expression can be obtained by invoking the `factor()` method on the coordinate function representing the scalar field in the manifold's default chart:

In [38]:
```
Kr = Kr_scalar.coord_function()
Kr.factor()
```

Out[38]:

$$-\frac{48 \left(a^2 \cos(\theta)^2 + 4\, ar\cos(\theta) + r^2\right)\left(a^2 \cos(\theta)^2 - 4\, ar\cos(\theta) + r^2\right)(a\cos(\theta) + r)(a\cos(}{\left(a^2 \cos(\theta)^2 + r^2\right)^6}$$

As a check, we can compare Kr to the formula given by R. Conn Henry, Astrophys. J. **535**, 350 (2000):

In [39]:
```
Kr == 48*m^2*(r^6 - 15*r^4*(a*cos(th))^2 + 15*r^2*(a*cos(th))^4
              - (a*cos(th))^6) / (r^2+(a*cos(th))^2)^6
```

Out[39]: True

The Schwarzschild value of the Kretschmann scalar is recovered by setting $a = 0$:

In [40]:
```
Kr.expr().subs(a=0)
```

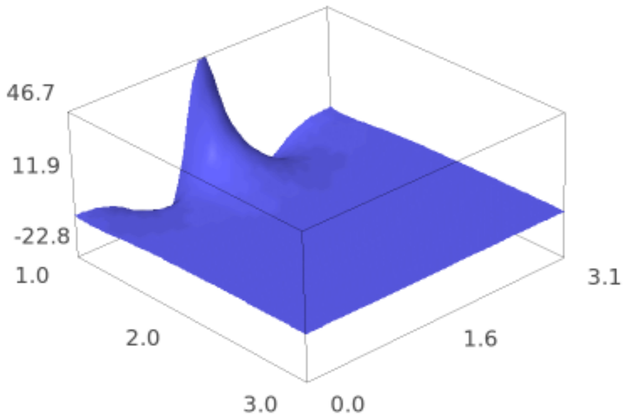
Out[40]:

$$\frac{48\, m^2}{r^6}$$

Let us plot the Kretschmann scalar for $m = 1$ and $a = 0.9$:

In [41]:
```
K1 = Kr.expr().subs(m=1, a=0.9)
plot3d(K1, (r,1,3), (th, 0, pi), viewer=viewer3D, axes_labels=['r', 'th
eta', 'Kr'])
```

Out[41]:



In [ ]: