3+1 slicing of Kerr spacetime

This worksheet demonstrates a few capabilities of SageManifolds (version 1.0, as included in SageMath 7.5) in computations regarding the 3+1 slicing of Kerr spacetime.

Click here to download the worksheet file (ipynb format). To run it, you must start SageMath within the Jupyter notebook, via the command `sage -n jupyter`

NB: a version of SageMath at least equal to 7.5 is required to run this worksheet:

In [1]:
```
version()
```

Out[1]:
```
'SageMath version 7.5.1, Release Date: 2017-01-15'
```

First we set up the notebook to display mathematical objects using LaTeX rendering:

In [2]:
```
%display latex
```

Since some computations are quite long, we ask for running them in parallel on 8 cores:

In [3]:
```
Parallelism().set(nproc=8)
```

Spacelike hypersurface

We consider some hypersurface Σ of a spacelike foliation $(\Sigma_t)_{t\in\mathbb{R}}$ of Kerr spacetime; we declare Σ_t as a 3-dimensional manifold:

In [4]:
```
Sig = Manifold(3, 'Sigma', r'\Sigma', start_index=1)
print(Sig)
```

```
3-dimensional differentiable manifold Sigma
```

On Σ, we consider the "rational-polynomial" coordinates (r, y, ϕ) inheritated from the standard Boyer-Lindquist coordinates (t, r, θ, ϕ) of Kerr spacetime, via $y = \cos\theta$:

In [5]:
```
X.<r,y,ph> = Sig.chart(r'r:(1,+oo) y:(-1,1) ph:(0,2*pi):\phi')
print(X) ; X
```

```
Chart (Sigma, (r, y, ph))
```

Out[5]: $(\Sigma, (r, y, \phi))$

Riemannian metric on Σ

First the two Kerr parameters:

In [6]:
```
var('m, a', domain='real')
assume(m>0)
assume(a>0)
assumptions()
```

Out[6]:
$$\left[\texttt{r is real}, r > 1, \texttt{y is real}, y > (-1), y < 1, \texttt{ph is real}, \phi > 0, \phi < 2\pi, \texttt{m is real}, \texttt{a is real}, m > 0, a > 0\right]$$

For dealing with extreme Kerr, the following must be uncommented:

In [7]:
```
# m = 1 ; a = 1
```

Some shortcut notations:

In [8]:
```
rho2 = r^2 + a^2*y^2
Del = r^2 -2*m*r + a^2
AA2 = rho2*(r^2 + a^2) + 2*a^2*m*r*(1-y^2)
BB2 = r^2 + a^2 + 2*a^2*m*r*(1-y^2)/rho2
```

The metric γ induced by the spacetime metric g on Σ:

In [9]:
```
gam = Sig.riemannian_metric('gam', latex_name=r'\gamma')
gam[1,1] = rho2/Del
gam[2,2] = rho2/(1-y^2)
gam[3,3] = BB2*(1-y^2)
gam.display()
```

Out[9]:

$$\gamma = \left(\frac{a^2 y^2 + r^2}{a^2 - 2\,mr + r^2}\right) \mathrm{d}r \otimes \mathrm{d}r + \left(-\frac{a^2 y^2 + r^2}{y^2 - 1}\right) \mathrm{d}y \otimes \mathrm{d}y + \left(\frac{2\,\left(y^2 - 1\right) a^2 mr}{a^2 y^2 + r^2} - a^2 - r^2\right)\left(y^2 - 1\right) \mathrm{d}\phi \otimes \mathrm{d}\phi$$

A matrix view of the components w.r.t. coordinates (r, y, ϕ):

In [10]:
```
gam[:]
```

Out[10]:

$$\begin{pmatrix} \frac{a^2y^2+r^2}{a^2-2\,mr+r^2} & 0 & 0 \\ 0 & -\frac{a^2y^2+r^2}{y^2-1} & 0 \\ 0 & 0 & \left(\frac{2\,\left(y^2-1\right)a^2mr}{a^2y^2+r^2} - a^2 - r^2\right)\left(y^2 - 1\right) \end{pmatrix}$$

Lapse function and shift vector

In [11]:
```
N = Sig.scalar_field(sqrt(Del / BB2), name='N')
print(N)
N.display()
```

```
Scalar field N on the 3-dimensional differentiable manifold Sigma
```

Out[11]:

$$\begin{array}{llcl} N: & \Sigma & \longrightarrow & \mathbb{R} \\ & (r, y, \phi) & \longmapsto & \sqrt{-\frac{a^2-2\,mr+r^2}{\frac{2\,\left(y^2-1\right)a^2mr}{a^2y^2+r^2}-a^2-r^2}} \end{array}$$

In [12]:
```
b = Sig.vector_field('beta', latex_name=r'\beta')
b[3] = -2*m*r*a/AA2
# unset components are zero
b.display()
```

Out[12]:

$$\beta = \left(\frac{2\,amr}{2\,\left(y^2 - 1\right)a^2 mr - \left(a^2 y^2 + r^2\right)\left(a^2 + r^2\right)}\right) \frac{\partial}{\partial \phi}$$

Extrinsic curvature of Σ

We use the formula

$$K_{ij} = \frac{1}{2N}\mathcal{L}_\beta \gamma_{ij}$$

which is valid for any stationary spacetime:

In [13]:
```
K = gam.lie_der(b) / (2*N)
K.set_name('K')
print(K) ; K.display()
```

```
Field of symmetric bilinear forms K on the 3-dimensional differentiable
 manifold Sigma
```

Out[13]:

$$\left(\frac{\left(a^3mr^2 + 3\,amr^4 + \left(a^5m - a^3mr^2\right)y^4 - \left(a^5m + 3\,amr^4\right)y^2\right)\sqrt{2\,a^2mr + a^2r^2 + r^4 \cdots}}{\left(2\,a^2mr^3 + a^2r^4 + r^6 + \left(a^6 - 2\,a^4mr + a^4r^2\right)y^4 + 2\sqrt{a^2y^2 + r^2}\sqrt{a^2 \cdots} \quad \left(a^4mr + a^4r^2 - a^2mr^3 + a^2r^4\right)y^2\right)} \right.$$

Check (comparison with known formulas):

In [14]:
```
Krp = a*m*(1-y^2)*(3*r^4+a^2*r^2+a^2*(r^2-a^2)*y^2) / rho2^2/sqrt(Del*B
B2)
Krp
```

Out[14]:

$$\frac{\left(\left(a^2 - r^2\right)a^2y^2 - a^2r^2 - 3\,r^4\right)\left(y^2 - 1\right)am}{\left(a^2y^2 + r^2\right)^2\sqrt{-\left(\frac{2\,\left(y^2-1\right)a^2mr}{a^2y^2+r^2} - a^2 - r^2\right)\left(a^2 - 2\,mr + r^2\right)}}$$

In [15]:
```
K[1,3] - Krp
```

Out[15]: 0

In [16]:
```
Kyp = 2*m*r*a^3*(1-y^2)*y*sqrt(Del)/rho2^2/sqrt(BB2)
Kyp
```

Out[16]:

$$-\frac{2\,\sqrt{a^2 - 2\,mr + r^2}\left(y^2 - 1\right)a^3mry}{\sqrt{-\frac{2\,\left(y^2-1\right)a^2mr}{a^2y^2+r^2} + a^2 + r^2}\left(a^2y^2 + r^2\right)^2}$$

In [17]:
```
K[2,3] - Kyp
```

Out[17]: 0

For now on, we use the expressions Krp and Kyp above for $K_{r\phi}$ and K_{ry}, respectively:

In [18]:
```
K1 = Sig.sym_bilin_form_field('K')
K1[1,3] = Krp
K1[2,3] = Kyp
K = K1
K.display()
```

Out[18]:

$$K = \left(\frac{\left(\left(a^2 - r^2\right)a^2y^2 - a^2r^2 - 3\,r^4\right)\left(y^2 - 1\right)am}{\left(a^2y^2 + r^2\right)^2\sqrt{-\left(\frac{2\,(y^2-1)a^2mr}{a^2y^2+r^2} - a^2 - r^2\right)\left(a^2 - 2\,mr + r^2\right)}} \right) \mathrm{d}r \otimes \mathrm{d}\phi + \left(-\frac{2\,\sqrt{a^2 - 2\,mr + r^2}\left(y^2 - 1\right)a^3mry}{\sqrt{-\frac{2\,(y^2-1)a^2mr}{a^2y^2+r^2} + a^2 + r^2}\left(a^2y^2 + r^2\right)^2} \right) \mathrm{d}y \otimes \mathrm{d}\phi + \left(\frac{\left(\left(a^2 - r^2\right)a^2y^2 - a^2r^2 - 3\,r^4\right)\left(y^2 - 1\right)am}{\left(a^2y^2 + r^2\right)^2\sqrt{-\left(\frac{2\,(y^2-1)a^2mr}{a^2y^2+r^2} - a^2 - r^2\right)\left(a^2 - 2\,mr + r^2\right)}} \right) \mathrm{d}\phi \otimes \mathrm{d}r + \left(-\frac{2\,\sqrt{a^2 - 2\,mr + r^2}\left(y^2 - 1\right)a^3mry}{\sqrt{-\frac{2\,(y^2-1)a^2mr}{a^2y^2+r^2} + a^2 + r^2}\left(a^2y^2 + r^2\right)^2} \right) \mathrm{d}\phi \otimes \mathrm{d}y$$

In [19]:
```
K.display_comp()
```

Out[19]:

$$K_{r\,\phi} = \frac{\left(\left(a^2-r^2\right)a^2y^2-a^2r^2-3\,r^4\right)\left(y^2-1\right)am}{\left(a^2y^2+r^2\right)^2\sqrt{-\left(\frac{2\,(y^2-1)a^2mr}{a^2y^2+r^2}-a^2-r^2\right)\left(a^2-2\,mr+r^2\right)}}$$

$$K_{y\,\phi} = -\frac{2\,\sqrt{a^2-2\,mr+r^2}\left(y^2-1\right)a^3mry}{\sqrt{-\frac{2\,(y^2-1)a^2mr}{a^2y^2+r^2}+a^2+r^2}\left(a^2y^2+r^2\right)^2}$$

$$K_{\phi\,r} = \frac{\left(\left(a^2-r^2\right)a^2y^2-a^2r^2-3\,r^4\right)\left(y^2-1\right)am}{\left(a^2y^2+r^2\right)^2\sqrt{-\left(\frac{2\,(y^2-1)a^2mr}{a^2y^2+r^2}-a^2-r^2\right)\left(a^2-2\,mr+r^2\right)}}$$

$$K_{\phi\,y} = -\frac{2\,\sqrt{a^2-2\,mr+r^2}\left(y^2-1\right)a^3mry}{\sqrt{-\frac{2\,(y^2-1)a^2mr}{a^2y^2+r^2}+a^2+r^2}\left(a^2y^2+r^2\right)^2}$$

The type-(1,1) tensor $K^\sharp$ of components $K^i{}_j = \gamma^{ik} K_{kj}$:

In [20]:
```
Ku = K.up(gam, 0)
print(Ku) ; Ku.display()
```

```
Tensor field of type (1,1) on the 3-dimensional differentiable manifold
 Sigma
```

Out[20]:

$$\Bigg(\frac{}{\left(a^6 y^6 + 3\right.}$$

$$+ \Bigg(\frac{}{\left(a^6\right.}$$

$$\Bigg(\frac{\left(a^3 m r^2 + 3\, a m r^4 - \left(a^5 m - a^3 m r^2\right) y^2\right) \sqrt{a^2 y^2 + r}}{\left(2\, a^2 m r^3 + a^2 r^4 + r^6 + \left(a^6 - 2\, a^4 m r + a^4 r^2\right) y^4 + 2 \sqrt{2\, a^2 m r + a^2 r^2 + r^4 + \left(a^4 - \right.}\right.} \left(a^4 m r + a^4 r^2 - a^2 m r^3 + a^2 r^4\right) y^2 \Big)$$

We may check that the hypersurface Σ is maximal, i.e. that $K^k_{\ \ k} = 0$:

In [21]:
```
trK = Ku.trace()
print(trK)
trK.display()
```

```
Scalar field zero on the 3-dimensional differentiable manifold Sigma
```

Out[21]:

$$\begin{array}{llcl} 0: & \Sigma & \longrightarrow & \mathbb{R} \\ & (r, y, \phi) & \longmapsto & 0 \end{array}$$

Connection and curvature

Let us call D the Levi-Civita connection associated with γ:

In [22]:
```
D = gam.connection(name='D')
print(D) ; D
```

```
Levi-Civita connection D associated with the Riemannian metric gam on t
he 3-dimensional differentiable manifold Sigma
```

Out[22]: D

The Ricci tensor associated with γ:

In [23]:
```
Ric = gam.ricci()
print(Ric) ; Ric
```

```
Field of symmetric bilinear forms Ric(gam) on the 3-dimensional differe
ntiable manifold Sigma
```

Out[23]: $\mathrm{Ric}\left(\gamma\right)$

In [24]:
```
Ric.display_comp(only_nonredundant=True)
```

Out[24]:
```
\newcommand{\Bold}[1]{\mathbf{#1}}\begin{array}{lcl} \mathrm{Ric}\left(\gamma\right)_{ \, r \, r }^{ \phantom{\, r}\ph
```

In [25]:
```
Ric[1,1]
```

Out[25]:

$$-\frac{\begin{aligned}&5\,a^{6}m^{2}r^{4}+2\,a^{4}m^{2}r^{6}+8\,a^{4}mr^{7}-7\,a^{2}m^{2}r^{8}+7\,a^{2}mr^{9}+2\,mr^{11}\\&+\left(a^{10}m^{2}+3\,a^{10}mr-14\,a^{8}m^{2}r^{2}-11\,a^{6}m^{2}r^{4}+3\,a^{6}mr^{5}+6\,\left(a^{8}m+2\,a^{6}m^{3}\right)r^{3}\right)y^{6}\\&+\left(3\,a^{6}m-4\,a^{4}m^{3}\right)r^{5}\\&-\left(a^{10}m^{2}+9\,a^{10}mr-30\,a^{8}m^{2}r^{2}-35\,a^{6}m^{2}r^{4}-16\,a^{4}m^{2}r^{6}+4\,a^{4}mr^{7}y^{4}\right.\\&\quad\left.+\left(17\,a^{6}m+4\,a^{4}m^{3}\right)r^{5}+2\,\left(11\,a^{8}m+12\,a^{6}m^{3}\right)r^{3}\right)\\&-\left(16\,a^{8}m^{2}r^{2}+29\,a^{6}m^{2}r^{4}+18\,a^{4}m^{2}r^{6}+16\,a^{4}mr^{7}-7\,a^{2}m^{2}r^{8}+5\,a^{2}mr^{9}y^{2}\right.\\&\quad\left.+\left(17\,a^{6}m-8\,a^{4}m^{3}\right)r^{5}+6\,\left(a^{8}m-2\,a^{6}m^{3}\right)r^{3}\right)\end{aligned}}{\begin{aligned}&4\,a^{6}m^{2}r^{6}+6\,a^{4}mr^{9}+3\,a^{2}r^{12}-2\,mr^{13}+r^{14}+\left(3\,a^{4}-8\,a^{2}m^{2}\right)r^{10}\\&+\left(a^{6}-4\,a^{4}m^{2}\right)r^{8}\\&+\left(a^{14}-6\,a^{12}mr-6\,a^{8}mr^{5}+a^{8}r^{6}+3\,\left(a^{10}+4\,a^{8}m^{2}\right)r^{4}-4y^{8}+4\right.\\&\quad\left.\left(3\,a^{10}m+2\,a^{8}m^{3}\right)r^{3}+3\,\left(a^{12}+4\,a^{10}m^{2}\right)r^{2}\right)\\&\left(a^{6}m-2\,a^{4}m^{3}\right)r^{7}+4\\&\left(a^{12}mr-5\,a^{6}mr^{7}+a^{6}r^{8}+\left(3\,a^{8}+8\,a^{6}m^{2}\right)r^{6}-\left(9\,a^{8}m+4\,a^{6}m^{3}\right)r^{5}y^{6}+2\right.\\&\left.+\left(3\,a^{10}+4\,a^{8}m^{2}\right)r^{4}-\left(3\,a^{10}m-4\,a^{8}m^{3}\right)r^{3}+\left(a^{12}-4\,a^{10}m^{2}\right)r^{2}\right)\\&\left(2\,a^{10}m^{2}r^{2}+16\,a^{6}m^{3}r^{5}-12\,a^{4}mr^{9}+3\,a^{4}r^{10}+\left(9\,a^{6}+14\,a^{4}m^{2}\right)r^{8}-2y^{4}+4\right.\\&\left(9\,a^{6}m+2\,a^{4}m^{3}\right)r^{7}+3\,\left(3\,a^{8}-2\,a^{6}m^{2}\right)r^{6}+3\,\left(a^{10}-6\,a^{8}m^{2}\right)r^{4}+2\\&\left.\left(3\,a^{10}m-2\,a^{8}m^{3}\right)r^{3}\right)\\&\left(2\,a^{8}m^{2}r^{4}-3\,a^{4}mr^{9}-3\,a^{2}mr^{11}+a^{2}r^{12}+\left(3\,a^{4}+2\,a^{2}m^{2}\right)r^{10}+3y^{2}\right.\\&\left(a^{6}-2\,a^{4}m^{2}\right)r^{8}+\left(3\,a^{6}m+4\,a^{4}m^{3}\right)r^{7}+\left(a^{8}-6\,a^{6}m^{2}\right)r^{6}\\&\left.+\left(3\,a^{8}m-4\,a^{6}m^{3}\right)r^{5}\right)\end{aligned}}$$

In [26]:
```
Ric[1,2]
```

Out[26]:

$$\frac{\left(3\,a^{10}m - 4\,a^{8}m^{2}r + 6\,a^{8}mr^{2} - 8\,a^{6}m^{2}r^{3} + 3\,a^{6}mr^{4}\right)y^{5} + 2\left(2\,a^{8}m^{2}r - 3\,a^{8}mr^{2} + 12\,a^{6}m^{2}r^{3} - 6\,a^{6}mr^{4} + 6\,a^{4}m^{2}r^{5} - 3\,a^{4}mr^{6}\right)y^{3} - \left(16\,a^{6}m^{2}r^{3} + 9\,a^{6}mr^{4} + 12\,a^{4}m^{2}r^{5} + 18\,a^{4}mr^{6} + 9\,a^{2}mr^{8}\right)y}{4\,a^{4}m^{2}r^{6} + 4\,a^{4}mr^{7} + a^{4}r^{8} + 4\,a^{2}mr^{9} + 2\,a^{2}r^{10} + r^{12} + \left(a^{12} - 4\,a^{10}mr - 4\,a^{8}mr^{3} + a^{8}r^{4} + 2\left(a^{10} + 2\,a^{8}m^{2}\right)r^{2}\right)y^{8} + 4\left(a^{10}mr - 2\,a^{8}mr^{3} - 3\,a^{6}mr^{5} + a^{6}r^{6} + 2\left(a^{8} + a^{6}m^{2}\right)r^{4} + \left(a^{10} - 2\,a^{8}m^{2}\right)r^{2}\right)y^{6} + 2\left(2\,a^{8}m^{2}r^{2} + 6\,a^{8}mr^{3} - 6\,a^{4}mr^{7} + 3\,a^{4}r^{8} + 2\left(3\,a^{6} + a^{4}m^{2}\right)r^{6} + \left(3\,a^{8} - 8\,a^{6}m^{2}\right)r^{4}\right)y^{4} + 4\left(2\,a^{6}m^{2}r^{4} + 3\,a^{6}mr^{5} + 2\,a^{4}mr^{7} + 2\,a^{4}r^{8} - a^{2}mr^{9} + a^{2}r^{10} + \left(a^{6} - 2\,a^{4}m^{2}\right)r^{6}\right)y^{2}}$$

In [27]:
```
Ric[1,3]
```

Out[27]: 0

In [28]:
```
Ric[2,2]
```

Out[28]:

$$\frac{6\,a^{6}m^{2}r^{4} + 4\,a^{4}m^{2}r^{6} + 7\,a^{4}mr^{7} - 2\,a^{2}m^{2}r^{8} + 5\,a^{2}mr^{9} + mr^{11} + 2\left(3\,a^{10}mr - 10\,a^{8}m^{2}r^{2} - 10\,a^{6}m^{2}r^{4} + 3\,a^{6}mr^{5} + 2\left(3\,a^{8}m + 4\,a^{6}m^{3}\right)r^{3}\right)y^{6} + \left(3\,a^{6}m - 8\,a^{4}m^{3}\right)r^{5} - \left(9\,a^{10}mr - 34\,a^{8}m^{2}r^{2} - 36\,a^{6}m^{2}r^{4} - 2\,a^{4}m^{2}r^{6} - a^{4}mr^{7} + \left(7\,a^{6}m + 8\,a^{4}m^{3}\right)r^{5}y^{4} + \left(17\,a^{8}m + 32\,a^{6}m^{3}\right)r^{3}\right) - 2\left(7\,a^{8}m^{2}r^{2} + 11\,a^{6}m^{2}r^{4} + 3\,a^{4}m^{2}r^{6} + 7\,a^{4}mr^{7} - a^{2}m^{2}r^{8} + 2\,a^{2}mr^{9} + 8y^{2}\left(a^{6}m - a^{4}m^{3}\right)r^{5} + \left(3\,a^{8}m - 8\,a^{6}m^{3}\right)r^{3}\right)}{4\,a^{4}m^{2}r^{6} + 4\,a^{4}mr^{7} + a^{4}r^{8} + 4\,a^{2}mr^{9} + 2\,a^{2}r^{10} + r^{12} - \left(a^{12} - 4\,a^{10}mr - 4\,a^{8}mr^{3} + a^{8}r^{4} + 2\left(a^{10} + 2\,a^{8}m^{2}\right)r^{2}\right)y^{10} + \left(a^{12} - 8\,a^{10}mr + 4\,a^{8}mr^{3} + 12\,a^{6}mr^{5} - 4\,a^{6}r^{6} - \left(7\,a^{8} + 8\,a^{6}m^{2}\right)r^{4} - 2y^{8}\left(a^{10} - 6\,a^{8}m^{2}\right)r^{2}\right) + 2\left(2\,a^{10}mr - 10\,a^{8}mr^{3} - 6\,a^{6}mr^{5} + 6\,a^{4}mr^{7} - 3\,a^{4}r^{8} - 2\left(2\,a^{6} + a^{4}m^{2}\right)r^{6}y^{6} + \left(a^{8} + 12\,a^{6}m^{2}\right)r^{4} + 2\left(a^{10} - 3\,a^{8}m^{2}\right)r^{2}\right) + 2\left(2\,a^{8}m^{2}r^{2} + 6\,a^{8}mr^{3} - 6\,a^{6}mr^{5} - 10\,a^{4}mr^{7} - a^{4}r^{8} + 2\,a^{2}mr^{9} - 2\,a^{2}r^{10} + 2y^{4}\left(2\,a^{6} + 3\,a^{4}m^{2}\right)r^{6} + 3\left(a^{8} - 4\,a^{6}m^{2}\right)r^{4}\right) + \left(8\,a^{6}m^{2}r^{4} + 12\,a^{6}mr^{5} + 4\,a^{4}mr^{7} + 7\,a^{4}r^{8} - 8\,a^{2}mr^{9} + 2\,a^{2}r^{10} - r^{12} + 4y^{2}\left(a^{6} - 3\,a^{4}m^{2}\right)r^{6}\right)}$$

In [29]:
```
Ric[2,3]
```

Out[29]: 0

In [30]:
```
Ric[3,3]
```

Out[30]:

$$\frac{\begin{array}{l} a^{6} m^{2} r^{4} + 4\, a^{4} m^{3} r^{5} + 10\, a^{4} m^{2} r^{6} + a^{4} m r^{7} + 13\, a^{2} m^{2} r^{8} + 2\, a^{2} m r^{9} + m r^{11} \\ + \left(a^{10} m^{2} + 3\, a^{10} m r - 18\, a^{8} m^{2} r^{2} - 15\, a^{6} m^{2} r^{4} + 3\, a^{6} m r^{5} + 2 y^{8} \right. \\ \quad \left. \left(3\, a^{8} m + 10\, a^{6} m^{3}\right) r^{3}\right) \\ - \left(2\, a^{10} m^{2} + 3\, a^{10} m r - 38\, a^{8} m^{2} r^{2} - 22\, a^{6} m^{2} r^{4} + 2\, a^{4} m^{2} r^{6} - 5\, a^{4} m r^{7} y^{6} \right. \\ \quad \left. - \left(7\, a^{6} m - 4\, a^{4} m^{3}\right) r^{5} + \left(a^{8} m + 60\, a^{6} m^{3}\right) r^{3}\right) \\ + \left(a^{10} m^{2} - 22\, a^{8} m^{2} r^{2} + 2\, a^{6} m^{2} r^{4} + 14\, a^{4} m^{2} r^{6} - 3\, a^{4} m r^{7} + 13\, a^{2} m^{2} r^{8} + a^{2} m r^{9} y^{4} \right. \\ \quad \left. - 3\, \left(3\, a^{6} m - 4\, a^{4} m^{3}\right) r^{5} - 5\, \left(a^{8} m - 12\, a^{6} m^{3}\right) r^{3}\right) \\ + \left(2\, a^{8} m^{2} r^{2} - 20\, a^{6} m^{3} r^{3} - 10\, a^{6} m^{2} r^{4} - 22\, a^{4} m^{2} r^{6} - 3\, a^{4} m r^{7} - 26\, a^{2} m^{2} r^{8} - 3 y^{2} \right. \\ \quad \left. a^{2} m r^{9} - m r^{11} - \left(a^{6} m + 12\, a^{4} m^{3}\right) r^{5}\right) \end{array}}{\begin{array}{l} 2\, a^{2} m r^{9} + a^{2} r^{10} + r^{12} + \left(a^{12} - 2\, a^{10} m r + a^{10} r^{2}\right) y^{10} \\ + \left(2\, a^{10} m r + 5\, a^{10} r^{2} - 8\, a^{8} m r^{3} + 5\, a^{8} r^{4}\right) y^{8} + 2 \\ \left(4\, a^{8} m r^{3} + 5\, a^{8} r^{4} - 6\, a^{6} m r^{5} + 5\, a^{6} r^{6}\right) y^{6} + 2 \\ \left(6\, a^{6} m r^{5} + 5\, a^{6} r^{6} - 4\, a^{4} m r^{7} + 5\, a^{4} r^{8}\right) y^{4} \\ + \left(8\, a^{4} m r^{7} + 5\, a^{4} r^{8} - 2\, a^{2} m r^{9} + 5\, a^{2} r^{10}\right) y^{2} \end{array}}$$

The scalar curvature $R = \gamma^{ij} R_{ij}$:

In [31]:
```
R = gam.ricci_scalar(name='R')
print(R)
R.display()
```

```
Scalar field R on the 3-dimensional differentiable manifold Sigma
```

Out[31]:

$$\begin{array}{llcl} \mathrm{r}\left(\gamma\right): & \Sigma & \longrightarrow & \mathbb{R} \\ & \left(r, y, \phi\right) & \longmapsto & \dfrac{\begin{array}{l} 2\, \left(a^{6} m^{2} r^{4} + 6\, a^{4} m^{2} r^{6} + 9\, a^{2} m^{2} r^{8} - \left(a^{10} m^{2} - 6\, a^{8} m^{2} r^{2} + 8\, a^{6} m^{3} r^{3} - 3\, a^{6} m^{2} r^{4}\right) y^{6} \right. \\ \quad + \left(a^{10} m^{2} - 8\, a^{8} m^{2} r^{2} + 16\, a^{6} m^{3} r^{3} - 3\, a^{6} m^{2} r^{4} - 6\, a^{4} m^{2} r^{6}\right) y^{4} + \left(2\, a^{8} m^{2} r^{2} - 8\, a^{6} m^{3} r^{3} \right. \end{array}}{\begin{array}{l} 4\, a^{4} m^{2} r^{8} + 4\, a^{4} m r^{9} + a^{4} r^{10} + 4\, a^{2} m r^{11} + 2\, a^{2} r^{12} + r^{14} + \left(a^{14} - 4\, a^{12} m r - 4\, a^{10} m r^{3} + a^{10} r \right. \\ + \left(4\, a^{12} m r - 12\, a^{10} m r^{3} - 16\, a^{8} m r^{5} + 5\, a^{8} r^{6} + 2\, \left(5\, a^{10} + 6\, a^{8} m^{2}\right) r^{4} + \left(5\, a^{12} - 8\, a^{10} m^{2}\right) r \right. \\ \left(2\, a^{10} m^{2} r^{2} + 8\, a^{10} m r^{3} - 4\, a^{8} m r^{5} - 12\, a^{6} m r^{7} + 5\, a^{6} r^{8} + 2\, \left(5\, a^{8} + 3\, a^{6} m^{2}\right) r^{6} + \left(5\, a^{10} - 1 \right. \right. \\ \left(6\, a^{8} m^{2} r^{4} + 12\, a^{8} m r^{5} + 4\, a^{6} m r^{7} - 8\, a^{4} m r^{9} + 5\, a^{4} r^{10} + 2\, \left(5\, a^{6} + a^{4} m^{2}\right) r^{8} + \left(5\, a^{8} - 12\, a \right. \right. \\ + \left(12\, a^{6} m^{2} r^{6} + 16\, a^{6} m r^{7} + 12\, a^{4} m r^{9} + 10\, a^{4} r^{10} - 4\, a^{2} m r^{11} + 5\, a^{2} r^{12} + \left(5\, a^{6} - 8\, a^{4} m^{2}\right) r \right. \end{array}} \end{array}$$

3+1 Einstein equations

Let us check that the vacuum 3+1 Einstein equations are satisfied.

We start by the contraint equations:

Hamiltonian constraint

Let us first evaluate the term $K_{ij} K^{ij}$:

In [32]:
```
Kuu = Ku.up(gam, 1)
trKK = K['_ij']*Kuu['^ij']
print(trKK) ; trKK.display()
```

```
Scalar field on the 3-dimensional differentiable manifold Sigma
```

Out[32]:

$$\begin{array}{llcl} & \Sigma & \longrightarrow & \mathbb{R} \\ & (r, y, \phi) & \longmapsto & \frac{\begin{array}{l} 2\,(a^{6} m^{2} r^{4} + 6\,a^{4} m^{2} r^{6} + 9\,a^{2} m^{2} r^{8} - (a^{10} m^{2} - 6\,a^{8} m^{2} r^{2} + 8\,a^{6} m^{3} r^{3} - 3\,a^{6} m^{2} r^{4}) y^{6} \\ \quad + (a^{10} m^{2} - 8\,a^{8} m^{2} r^{2} + 16\,a^{6} m^{3} r^{3} - 3\,a^{6} m^{2} r^{4} - 6\,a^{4} m^{2} r^{6}) y^{4} + (2\,a^{8} m^{2} r^{2} - 8\,a^{6} m^{3} r^{3} - a^{6} m^{2} r \end{array}}{\begin{array}{l} 4\,a^{4} m^{2} r^{8} + 4\,a^{4} m r^{9} + a^{4} r^{10} + 4\,a^{2} m r^{11} + 2\,a^{2} r^{12} + r^{14} + (a^{14} - 4\,a^{12} m r - 4\,a^{10} m r^{3} + a^{10} r^{4} + 2\,(a^{1} \\ + (4\,a^{12} m r - 12\,a^{10} m r^{3} - 16\,a^{8} m r^{5} + 5\,a^{8} r^{6} + 2\,(5\,a^{10} + 6\,a^{8} m^{2}) r^{4} + (5\,a^{12} - 8\,a^{10} m^{2}) r^{2}) y^{8} + 2 \\ (2\,a^{10} m^{2} r^{2} + 8\,a^{10} m r^{3} - 4\,a^{8} m r^{5} - 12\,a^{6} m r^{7} + 5\,a^{6} r^{8} + 2\,(5\,a^{8} + 3\,a^{6} m^{2}) r^{6} + (5\,a^{10} - 12\,a^{8} m^{2} \\ (6\,a^{8} m^{2} r^{4} + 12\,a^{8} m r^{5} + 4\,a^{6} m r^{7} - 8\,a^{4} m r^{9} + 5\,a^{4} r^{10} + 2\,(5\,a^{6} + a^{4} m^{2}) r^{8} + (5\,a^{8} - 12\,a^{6} m^{2}) r^{6}) \\ + (12\,a^{6} m^{2} r^{6} + 16\,a^{6} m r^{7} + 12\,a^{4} m r^{9} + 10\,a^{4} r^{10} - 4\,a^{2} m r^{11} + 5\,a^{2} r^{12} + (5\,a^{6} - 8\,a^{4} m^{2}) r^{8}) y^{2} \end{array}} \end{array}$$

The vacuum Hamiltonian constraint equation is

$$R + K^2 - K_{ij} K^{ij} = 0$$

In [33]:
```
Ham = R + trK^2 - trKK
print(Ham) ; Ham.display()
```

```
Scalar field zero on the 3-dimensional differentiable manifold Sigma
```

Out[33]:

$$\begin{array}{llcl} 0: & \Sigma & \longrightarrow & \mathbb{R} \\ & (r, y, \phi) & \longmapsto & 0 \end{array}$$

Momentum constraint

In vaccum, the momentum constraint is

$$D_j K^j{}_i - D_i K = 0$$

In [34]:
```
mom = D(Ku).trace(0,2) - D(trK)
print(mom)
mom.display()
```

```
1-form on the 3-dimensional differentiable manifold Sigma
```

Out[34]: 0

Dynamical Einstein equations

Let us first evaluate the symmetric bilinear form $k_{ij} := K_{ik} K^k{}_j$:

In [35]:
```
KK = K['_ik']*Ku['^k_j']
print(KK)
```

```
Tensor field of type (0,2) on the 3-dimensional differentiable manifold
 Sigma
```

In [36]:
```
KK1 = KK.symmetrize()
KK == KK1
```

Out[36]: True

In [37]:
```
KK = KK1
print(KK)
```

```
Field of symmetric bilinear forms on the 3-dimensional differentiable m
anifold Sigma
```

In [38]:
```
KK.set_name('(KK)')
KK.display_comp()
```

Out[38]:

$$(KK)_{r\,r} = \frac{a^6m^2r^4+6\,a^4m^2r^6+9\,a^2m^2r^8-(a^{10}m^2-2\,a^8m^2r^2+a^6m^2r^4)y^6+(a^{10}m^2+5\,a^6m^2r^4-6\,a^4 \quad -(2\,a^8m^2r^2+5\,a^6m^2r^4+9\,a^2m^2r^8)y^2}{4\,a^6m^2r^6+6\,a^4mr^9+3\,a^2r^{12}-2\,mr^{13}+r^{14}+(3\,a^4-8\,a^2m^2)r^{10}+(a^6-4\,a^4m^2)r^8 +(a^{14}-6\,a^{12}mr-6\,a^8mr^5+a^8r^6+3\,(a^{10}+4\,a^8m^2)r^4-4\,(3\,a^{10}m+2\,a^8m^3)r^3+3\,(a^{12}+4\,a^{10}m^2)r^2 \quad (a^6m-2\,a^4m^3)r^7+4 \quad (a^{12}mr-5\,a^6mr^7+a^6r^8+(3\,a^8+8\,a^6m^2)r^6-(9\,a^8m+4\,a^6m^3)r^5+(3\,a^{10}+4\,a^8m^2)r^4-(3\,a^{10}m-4 \quad +(a^{12}-4\,a^{10}m^2)r^2) \quad (2\,a^{10}m^2r^2+16\,a^6m^3r^5-12\,a^4mr^9+3\,a^4r^{10}+(9\,a^6+14\,a^4m^2)r^8-2\,(9\,a^6m+2\,a^4m^3)r^7+3\,(3\,a^8 \quad (a^{10}-6\,a^8m^2)r^4+2\,(3\,a^{10}m-2\,a^8m^3)r^3) \quad +4\,(2\,a^8m^2r^4-3\,a^4mr^9-3\,a^2mr^{11}+a^2r^{12}+(3\,a^4+2\,a^2m^2)r^{10}+3\,(a^6-2\,a^4m^2)r^8+(3\,a^6m+4\,a^4 \quad +(a^8-6\,a^6m^2)r^6+(3\,a^8m-4\,a^6m^3)r^5)}$$

$$(KK)_{r\,y} = \frac{2\,((a^8m^2r-a^6m^2r^3)y^5-(a^8m^2r+3\,a^4m^2r^5)y^3+(a^6m^2r^3+3\,a^4m^2r^5)y)}{4\,a^4m^2r^6+4\,a^4mr^7+a^4r^8+4\,a^2mr^9+2\,a^2r^{10}+r^{12}+(a^{12}-4\,a^{10}mr-4\,a^8mr^3+a^8r^4+2\,(a^{10}+2\,a^8m \quad (a^{10}mr-2\,a^8mr^3-3\,a^6mr^5+a^6r^6+2\,(a^8+a^6m^2)r^4+(a^{10}-2\,a^8m^2)r^2)y^6+2 \quad (2\,a^8m^2r^2+6\,a^8mr^3-6\,a^4mr^7+3\,a^4r^8+2\,(3\,a^6+a^4m^2)r^6+(3\,a^8-8\,a^6m^2)r^4)y^4+4 \quad (2\,a^6m^2r^4+3\,a^6mr^5+2\,a^4mr^7+2\,a^4r^8-a^2mr^9+a^2r^{10}+(a^6-2\,a^4m^2)r^6)y^2}$$

$$(KK)_{y\,r} = \frac{2\,((a^8m^2r-a^6m^2r^3)y^5-(a^8m^2r+3\,a^4m^2r^5)y^3+(a^6m^2r^3+3\,a^4m^2r^5)y)}{4\,a^4m^2r^6+4\,a^4mr^7+a^4r^8+4\,a^2mr^9+2\,a^2r^{10}+r^{12}+(a^{12}-4\,a^{10}mr-4\,a^8mr^3+a^8r^4+2\,(a^{10}+2\,a^8m \quad (a^{10}mr-2\,a^8mr^3-3\,a^6mr^5+a^6r^6+2\,(a^8+a^6m^2)r^4+(a^{10}-2\,a^8m^2)r^2)y^6+2 \quad (2\,a^8m^2r^2+6\,a^8mr^3-6\,a^4mr^7+3\,a^4r^8+2\,(3\,a^6+a^4m^2)r^6+(3\,a^8-8\,a^6m^2)r^4)y^4+4 \quad (2\,a^6m^2r^4+3\,a^6mr^5+2\,a^4mr^7+2\,a^4r^8-a^2mr^9+a^2r^{10}+(a^6-2\,a^4m^2)r^6)y^2}$$

$$(KK)_{y\,y} = -\frac{4\,((a^8m^2r^2-2\,a^6m^3r^3+a^6m^2r^4)y^4-(a^8m^2r^2-2\,a^6m^3r^3+a^6m^2r^4)y^2)}{4\,a^4m^2r^6+4\,a^4mr^7+a^4r^8+4\,a^2mr^9+2\,a^2r^{10}+r^{12}+(a^{12}-4\,a^{10}mr-4\,a^8mr^3+a^8r^4+2\,(a^{10}+2\,a^8 \quad (a^{10}mr-2\,a^8mr^3-3\,a^6mr^5+a^6r^6+2\,(a^8+a^6m^2)r^4+(a^{10}-2\,a^8m^2)r^2)y^6+2 \quad (2\,a^8m^2r^2+6\,a^8mr^3-6\,a^4mr^7+3\,a^4r^8+2\,(3\,a^6+a^4m^2)r^6+(3\,a^8-8\,a^6m^2)r^4)y^4+4 \quad (2\,a^6m^2r^4+3\,a^6mr^5+2\,a^4mr^7+2\,a^4r^8-a^2mr^9+a^2r^{10}+(a^6-2\,a^4m^2)r^6)y^2}$$

$$(KK)_{\phi\,\phi} = \frac{a^6m^2r^4+6\,a^4m^2r^6+9\,a^2m^2r^8+(a^{10}m^2-6\,a^8m^2r^2+8\,a^6m^3r^3-3\,a^6m^2r^4)y^8-2\,(a^{10}m^2-7\,a^8m^2r^2+12\,a^6m^3r^3-3\,a^6m^2r^4-3\,a^4m^2r^6)y^6+(a^{10}m^2-10\,a^8m^2r^2+24\,a^6m^3r^3-2\,a^6m^2r^4-6\,a^4m^2r^6+9\,a^2m^2r^8)y^4+2\,(a^8m^2r^2-4\,a^6m^3r^3-a^6m^2r^4-3\,a^4m^2r^6-9\,a^2m^2r^8)y^2}{2\,a^2mr^9+a^2r^{10}+r^{12}+(a^{12}-2\,a^{10}mr+a^{10}r^2)y^{10}+(2\,a^{10}mr+5\,a^{10}r^2-8\,a^8mr^3+5\,a^8r^4)y^8+2\,(4\,a^8mr^3+5\,a^8r^4-6\,a^6mr^5+5\,a^6r^6)y^6+2\,(6\,a^6mr^5+5\,a^6r^6-4\,a^4mr^7+5\,a^4r^8)y^4+(8\,a^4mr^7+5\,a^4r^8-2\,a^2mr^9+5\,a^2r^{10})y^2}$$

In vacuum and for stationary spacetimes, the dynamical Einstein equations are

$$\mathcal{L}_\beta K_{ij} - D_iD_jN + N\left(R_{ij} + KK_{ij} - 2K_{ik}K^k_{\ j}\right) = 0$$

In [39]:
```
dyn = K.lie_der(b) - D(D(N)) + N*( Ric + trK*K - 2*KK )
print(dyn)
dyn.display()
```

```
Tensor field of type (0,2) on the 3-dimensional differentiable manifold
 Sigma
```

Out[39]: 0

Electric and magnetic parts of the Weyl tensor

The **electric part** is the bilinear form E given by

$$E_{ij} = R_{ij} + KK_{ij} - K_{ik}K^k_{\ \ j}$$

In [40]:
```
E = Ric + trK*K - KK
print(E)
```

```
Field of symmetric bilinear forms +Ric(gam)-(KK) on the 3-dimensional d
ifferentiable manifold Sigma
```

In [41]:
```
E.set_name('E')
E.display_comp(only_nonzero=False)
```

Out[41]:

$$E_{r\,r} = -\frac{3\,a^4mr^3 - 2\,a^2m^2r^4 + 5\,a^2mr^5 + 2\,mr^7 + 3\,\left(a^6mr - 2\,a^4m^2r^2 + a^4mr^3\right)y^4 - \left(9\,a^6mr - 6\,a^4m^2r^2 + 16\,a^4mr^3 - 2\,a^2m^2r^4 + 7\,a^2mr^5\right)y^2}{2\,a^4mr^5 + 2\,a^2r^8 - 2\,mr^9 + r^{10} + \left(a^4 - 4\,a^2m^2\right)r^6 + \left(a^{10} - 4\,a^8mr - 4\,a^6mr^3 + a^6r^4 + 2\,\left(a^8 + 2\,a^6m^2\right)r^2\right)y^6 + \left(2\,a^8mr - 8\,a^6mr^3 - 10\,a^4mr^5 + 3\,a^4r^6 + 2\,\left(3\,a^6 + 4\,a^4m^2\right)r^4 + \left(3\,a^8 - 4\,a^6m^2\right)r^2\right)y^4 + \left(4\,a^6mr^3 - 4\,a^4mr^5 - 8\,a^2mr^7 + 3\,a^2r^8 + 2\,\left(3\,a^4 + 2\,a^2m^2\right)r^6 + \left(3\,a^6 - 8\,a^4m^2\right)r^4\right)y^2}$$

$$E_{r\,y} = \frac{3\,\left(\left(a^6m + a^4mr^2\right)y^3 - 3\,\left(a^4mr^2 + a^2mr^4\right)y\right)}{2\,a^2mr^5 + a^2r^6 + r^8 + \left(a^8 - 2\,a^6mr + a^6r^2\right)y^6 + \left(2\,a^6mr + 3\,a^6r^2 - 4\,a^4mr^3 + 3\,a^4r^4\right)y^4 + \left(4\,a^4mr^3 + 3\,a^4r^4 - 2\,a^2mr^5 + 3\,a^2r^6\right)y^2}$$

$$E_{r\,\phi} = 0$$

$$E_{y\,r} = \frac{3\,\left(\left(a^6m + a^4mr^2\right)y^3 - 3\,\left(a^4mr^2 + a^2mr^4\right)y\right)}{2\,a^2mr^5 + a^2r^6 + r^8 + \left(a^8 - 2\,a^6mr + a^6r^2\right)y^6 + \left(2\,a^6mr + 3\,a^6r^2 - 4\,a^4mr^3 + 3\,a^4r^4\right)y^4 + \left(4\,a^4mr^3 + 3\,a^4r^4 - 2\,a^2mr^5 + 3\,a^2r^6\right)y^2}$$

$$E_{y\,y} = -\frac{3\,a^4mr^3 - 4\,a^2m^2r^4 + 4\,a^2mr^5 + mr^7 + 6\,\left(a^6mr - 2\,a^4m^2r^2 + a^4mr^3\right)y^4 - \left(9\,a^6mr - 12\,a^4m^2r^2 + 14\,a^4mr^3 - 4\,a^2m^2r^4 + 5\,a^2mr^5\right)y^2}{\left(a^8 - 2\,a^6mr + a^6r^2\right)y^8 - 2\,a^2mr^5 - a^2r^6 - r^8 - \left(a^8 - 4\,a^6mr - 2\,a^6r^2 + 4\,a^4mr^3 - 3\,a^4r^4\right)y^6 - \left(2\,a^6mr + 3\,a^6r^2 - 8\,a^4mr^3 + 2\,a^2mr^5 - 3\,a^2r^6\right)y^4 - \left(4\,a^4mr^3 + 3\,a^4r^4 - 4\,a^2mr^5 + 2\,a^2r^6 - r^8\right)y^2}$$

$$E_{y\,\phi} = 0$$

$$E_{\phi\,r} = 0$$

$$E_{\phi\,y} = 0$$

$$E_{\phi\,\phi} = \frac{2\,a^2m^2r^4 + a^2mr^5 + mr^7 + 3\,\left(a^6mr - 2\,a^4m^2r^2 + a^4mr^3\right)y^6 - \left(3\,a^6mr - 12\,a^4m^2r^2 + a^4mr^3 - 2\,a^2m^2r^4 - 2\,a^2 \right. - \left(6\,a^4m^2r^2 + 2\,a^4mr^3 + 4\,a^2m^2r^4 + 3\,a^2mr^5 + mr^7\right)y^2}{a^8y^8 + 4\,a^6r^2y^6 + 6\,a^4r^4y^4 + 4\,a^2r^6y^2 + r^8}$$

The **magnetic part** is the bilinear form B defined by

$$B_{ij} = \epsilon^k_{\ \ il} D_k K^l_{\ \ j},$$

where $\epsilon^k_{\ \ il}$ are the components of the type-(1,2) tensor $\epsilon^\sharp$, related to the Levi-Civita alternating tensor ϵ associated with γ by $\epsilon^k_{\ \ il} = \gamma^{km}\epsilon_{mil}$. In SageManifolds, ϵ is obtained by the command `volume_form()` and $\epsilon^\sharp$ by the command `volume_form(1)` (1 = one index raised):

In [42]:
```
eps = gam.volume_form()
print(eps) ; eps.display()
```

3-form eps_gam on the 3-dimensional differentiable manifold Sigma

Out[42]:

$$\epsilon_\gamma = \left(\frac{\sqrt{2\,a^2mr + a^2r^2 + r^4 + \left(a^4 - 2\,a^2mr + a^2r^2\right)y^2}\sqrt{a^2y^2 + r^2}}{\sqrt{a^2 - 2\,mr + r^2}}\right)\mathrm{d}r \wedge \mathrm{d}y \wedge \mathrm{d}\phi$$

In [43]:
```
epsu = gam.volume_form(1)
print(epsu) ; epsu.display()
```

Tensor field of type (1,2) on the 3-dimensional differentiable manifold Sigma

Out[43]:

$$\left(\frac{\sqrt{2\,a^2mr + a^2r^2 + r^4 + \left(a^4 - 2\,a^2mr + a^2r^2\right)y^2}\sqrt{a^2 - 2\,mr + r^2}}{\sqrt{a^2y^2 + r^2}}\right)\frac{\partial}{\partial r} \otimes$$

$$\otimes\, \mathrm{d}\phi + \left(-\frac{\sqrt{2\,a^2mr + a^2r^2 + r^4 + \left(a^4 - 2\,a^2mr + a^2r^2\right)y^2}\sqrt{a^2 - 2\,mr + r}}{\sqrt{a^2y^2 + r^2}}\right.$$

$$\otimes\, \mathrm{d}\phi \otimes \mathrm{d}y + \left(\frac{\sqrt{2\,a^2mr + a^2r^2 + r^4 + \left(a^4 - 2\,a^2mr + a^2r^2\right)y^2}\left(y^2 - 1\right)}{\sqrt{a^2y^2 + r^2}\sqrt{a^2 - 2\,mr + r^2}}\right.$$

$$\otimes\, \mathrm{d}r \otimes \mathrm{d}\phi + \left(-\frac{\sqrt{2\,a^2mr + a^2r^2 + r^4 + \left(a^4 - 2\,a^2mr + a^2r^2\right)y^2}\left(y^2 - 1\right)}{\sqrt{a^2y^2 + r^2}\sqrt{a^2 - 2\,mr + r^2}}\right.$$

$$\otimes\, \mathrm{d}\phi \otimes \mathrm{d}r$$

$$+ \left(-\frac{\sqrt{2\,a^2mr + a^2r^2 + r^4 + \left(a^4 - 2\,a^2mr + a^2r^2\right)y^2}\left(a^2y^2 + r^2\right)^{\frac{3}{2}}}{\left(\left(a^4 - 2\,a^2mr + a^2r^2\right)y^4 - 2\,a^2mr - a^2r^2 - r^4 - \left(a^4 - 4\,a^2mr - r^4\right)y^2\right)\sqrt{a^2 -}}\right.$$

$$\otimes\, \mathrm{d}r \otimes \mathrm{d}y$$

$$+ \left(\frac{\sqrt{2\,a^2mr + a^2r^2 + r^4 + \left(a^4 - 2\,a^2mr + a^2r^2\right)y^2}\left(a^2y^2 + r^2\right)^{\frac{3}{2}}}{\left(\left(a^4 - 2\,a^2mr + a^2r^2\right)y^4 - 2\,a^2mr - a^2r^2 - r^4 - \left(a^4 - 4\,a^2mr - r^4\right)y^2\right)\sqrt{a^2 -}}\right.$$

$$\otimes\, \mathrm{d}y \otimes \mathrm{d}r$$

In [44]:
```
DKu = D(Ku)
B = epsu['^k_il']*DKu['^l_jk']
print(B)
```

Tensor field of type (0,2) on the 3-dimensional differentiable manifold Sigma

Let us check that B is symmetric:

In [45]:
```
B1 = B.symmetrize()
B == B1
```

Out[45]: True

Accordingly, we set

In [46]:
```
B = B1
B.set_name('B')
print(B)
```

```
Field of symmetric bilinear forms B on the 3-dimensional differentiable
 manifold Sigma
```

In [47]:
```
B.display_comp(only_nonzero=False)
```

Out[47]:

$$B_{r\,r} = \frac{\left(a^7 m - 2\, a^5 m^2 r + a^5 m r^2\right) y^5 - \left(3\, a^7 m - 2\, a^5 m^2 r + 8\, a^5 m r^2 - 6\, a^3 m^2 r^3 + 5\, a^3 m r^4\right) y^3 + 3\, \left(3\, a^5 m r^2 - 2\, a^3 m^2 r^3 + 5\, a^3 m r^4 + 2\, a m r^6\right) y}{2\, a^4 m r^5 + 2\, a^2 r^8 - 2\, m r^9 + r^{10} + \left(a^4 - 4\, a^2 m^2\right) r^6 + \left(a^{10} - 4\, a^8 m r - 4\, a^6 m r^3 + a^6 r^4 + 2\, \left(a^8 + 2\, a^6 m^2\right) r^2\right) y^6 + \left(2\, a^8 m r - 8\, a^6 m r^3 - 10\, a^4 m r^5 + 3\, a^4 r^6 + 2\, \left(3\, a^6 + 4\, a^4 m^2\right) r^4 + \left(3\, a^8 - 4\, a^6 m^2\right) r^2\right) y^4 + \left(4\, a^6 m r^3 - 4\, a^4 m r^5 - 8\, a^2 m r^7 + 3\, a^2 r^8 + 2\, \left(3\, a^4 + 2\, a^2 m^2\right) r^6 + \left(3\, a^6 - 8\, a^4 m^2\right) r^4\right) y^2}$$

$$B_{r\,y} = -\frac{3\, \left(a^3 m r^3 + a m r^5 - 3\, \left(a^5 m r + a^3 m r^3\right) y^2\right)}{2\, a^2 m r^5 + a^2 r^6 + r^8 + \left(a^8 - 2\, a^6 m r + a^6 r^2\right) y^6 + \left(2\, a^6 m r + 3\, a^6 r^2 - 4\, a^4 m r^3 + 3\, a^4 r^4\right) y^4 + \left(4\, a^4 m r^3 + 3\, a^4 r^4 - 2\, a^2 m r^5 + 3\, a^2 r^6\right) y^2}$$

$$B_{r\,\phi} = 0$$

$$B_{y\,r} = -\frac{3\, \left(a^3 m r^3 + a m r^5 - 3\, \left(a^5 m r + a^3 m r^3\right) y^2\right)}{2\, a^2 m r^5 + a^2 r^6 + r^8 + \left(a^8 - 2\, a^6 m r + a^6 r^2\right) y^6 + \left(2\, a^6 m r + 3\, a^6 r^2 - 4\, a^4 m r^3 + 3\, a^4 r^4\right) y^4 + \left(4\, a^4 m r^3 + 3\, a^4 r^4 - 2\, a^2 m r^5 + 3\, a^2 r^6\right) y^2}$$

$$B_{y\,y} = \frac{2\, \left(a^7 m - 2\, a^5 m^2 r + a^5 m r^2\right) y^5 - \left(3\, a^7 m - 4\, a^5 m^2 r + 10\, a^5 m r^2 - 12\, a^3 m^2 r^3 + 7\, a^3 m r^4\right) y^3 + 3\, \left(3\, a^5 m r^2 - 4\, a^3 m^2 r^3 + 4\, a^3 m r^4 + a m r^6\right) y}{\left(a^8 - 2\, a^6 m r + a^6 r^2\right) y^8 - 2\, a^2 m r^5 - a^2 r^6 - r^8 - \left(a^8 - 4\, a^6 m r - 2\, a^6 r^2 + 4\, a^4 m r^3 - 3\, a^4 r^4\right) y^6 - \left(2\, a^6 m r + 3\, a^6 r^2 - 8\, a^4 m r^3 + 2\, a^2 m r^5 - 3\, a^2 r^6\right) y^4 - \left(4\, a^4 m r^3 + 3\, a^4 r^4 - 4\, a^2 m r^5 + 2\, a^2 r^6 - r^8\right) y^2}$$

$$B_{y\,\phi} = 0$$

$$B_{\phi\,r} = 0$$

$$B_{\phi\,y} = 0$$

$$B_{\phi\,\phi} = -\frac{\left(a^7 m - 2\, a^5 m^2 r + a^5 m r^2\right) y^7 - \left(a^7 m - 4\, a^5 m^2 r + 3\, a^5 m r^2 - 6\, a^3 m^2 r^3 + 2\, a^3 m r^4\right) y^5 - \left(2\, a^5 m^2 r - 2\, a^5 m r^2 + 12\, a^3 m^2 r^3 + a^3 m r^4 + 3\, a m r^6\right) y^3 + 3\, \left(2\, a^3 m^2 r^3 + a^3 m r^4 + a m r^6\right) y}{a^8 y^8 + 4\, a^6 r^2 y^6 + 6\, a^4 r^4 y^4 + 4\, a^2 r^6 y^2 + r^8}$$