Kerr-Newman spacetime

This worksheet demonstrates a few capabilities of SageManifolds (version 1.0, as included in SageMath 7.5) in computations regarding Kerr-Newman spacetime.

Click here to download the worksheet file (ipynb format). To run it, you must start SageMath within the Jupyter notebook, via the command `sage -n jupyter`

NB: a version of SageMath at least equal to 7.5 is required to run this worksheet:

In [1]:
```
version()
```

Out[1]: 'SageMath version 7.5, Release Date: 2017-01-11'

First we set up the notebook to display mathematical objects using LaTeX rendering:

In [2]:
```
%display latex
```

We also define a viewer for 3D plots (use `'threejs'` or `'jmol'` for interactive 3D graphics):

In [3]:
```
viewer3D = 'jmol' # must be 'threejs', jmol', 'tachyon' or None (defaul
t)
```

Since some computations are quite long, we ask for running them in parallel on 8 cores:

In [4]:
```
Parallelism().set(nproc=8)
```

Spacetime manifold

We declare the Kerr-Newman spacetime as a 4-dimensional diffentiable manifold:

In [5]:
```
M = Manifold(4, 'M', r'\mathcal{M}')
```

Let us use the standard **Boyer-Lindquist coordinates** on it, by first introducing the part $\mathcal{M}_0$ covered by these coordinates

In [6]:
```
M0 = M.open_subset('M0', r'\mathcal{M}_0')
# BL = Boyer-Lindquist
BL.<t,r,th,ph> = M0.chart(r't r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\p
hi')
print(BL) ; BL
```

```
Chart (M0, (t, r, th, ph))
```

Out[6]: $(\mathcal{M}_0, (t, r, \theta, \phi))$

Metric tensor

The 3 parameters m, a and q of the Kerr-Newman spacetime are declared as symbolic variables:

In [7]:
```
var('m a q')
```

Out[7]: (m, a, q)

Let us introduce the spacetime metric:

In [8]:
```
g = M.lorentzian_metric('g')
```

The metric is defined by its components in the coordinate frame associated with Boyer-Lindquist coordinates, which is the current manifold's default frame:

In [9]:
```
rho2 = r^2 + (a*cos(th))^2
Delta = r^2 -2*m*r + a^2 + q^2
g[0,0] = -1 + (2*m*r-q^2)/rho2
g[0,3] = -a*sin(th)^2*(2*m*r-q^2)/rho2
g[1,1], g[2,2] = rho2/Delta, rho2
g[3,3] = (r^2 + a^2 + (2*m*r-q^2)*(a*sin(th))^2/rho2)*sin(th)^2
g.display()
```

Out[9]:

$$g = \left(-\frac{q^2 - 2\,mr}{a^2 \cos(\theta)^2 + r^2} - 1\right) \mathrm{d}t \otimes \mathrm{d}t + \left(\frac{\left(q^2 - 2\,mr\right) a \sin(\theta)^2}{a^2 \cos(\theta)^2 + r^2}\right) \mathrm{d}t \otimes \mathrm{d}\phi + \left(\frac{a^2 \cos(\theta)^2 + r^2}{a^2 + q^2 - 2\,mr + r^2}\right) \mathrm{d}r \otimes \mathrm{d}r + \left(a^2 \cos(\theta)^2 + r^2\right) \mathrm{d}\theta \otimes \mathrm{d}\theta + \left(\frac{\left(q^2 - 2\,mr\right) a \sin(\theta)^2}{a^2 \cos(\theta)^2 + r^2}\right) \mathrm{d}\phi \otimes \mathrm{d}t - \left(\frac{\left(q^2 - 2\,mr\right) a^2 \sin(\theta)^2}{a^2 \cos(\theta)^2 + r^2} - a^2 - r^2\right) \sin(\theta)^2 \mathrm{d}\phi \otimes \mathrm{d}\phi$$

The list of the non-vanishing components:

In [10]:
```
g.display_comp()
```

Out[10]:

$$\begin{array}{lcl} g_{t\,t} & = & -\frac{q^2 - 2\,mr}{a^2 \cos(\theta)^2 + r^2} - 1 \\ g_{t\,\phi} & = & \frac{\left(q^2 - 2\,mr\right) a \sin(\theta)^2}{a^2 \cos(\theta)^2 + r^2} \\ g_{r\,r} & = & \frac{a^2 \cos(\theta)^2 + r^2}{a^2 + q^2 - 2\,mr + r^2} \\ g_{\theta\,\theta} & = & a^2 \cos(\theta)^2 + r^2 \\ g_{\phi\,t} & = & \frac{\left(q^2 - 2\,mr\right) a \sin(\theta)^2}{a^2 \cos(\theta)^2 + r^2} \\ g_{\phi\,\phi} & = & -\left(\frac{\left(q^2 - 2\,mr\right) a^2 \sin(\theta)^2}{a^2 \cos(\theta)^2 + r^2} - a^2 - r^2\right) \sin(\theta)^2 \end{array}$$

The component g^{tt} of the inverse metric:

In [11]:
```
g.inverse()[0,0]
```

Out[11]:

$$\frac{a^4 + 2\,a^2 r^2 + r^4 - \left(a^4 + a^2 q^2 - 2\,a^2 mr + a^2 r^2\right) \sin(\theta)^2}{2\,mr^3 - r^4 - \left(a^2 + q^2\right) r^2 - \left(a^4 + a^2 q^2 - 2\,a^2 mr + a^2 r^2\right) \cos(\theta)^2}$$

The lapse function:

In [12]:
```
N = 1/sqrt(-(g.inverse()[[0,0]])) ; N
```

Out[12]: Scalar field on the Open subset M0 of the 4-dimensional differentiable manifold M

In [13]:
```
N.display()
```

Out[13]:
$$\begin{array}{llcl} & \mathcal{M}_0 & \longrightarrow & \mathbb{R} \\ & (t,r,\theta,\phi) & \longmapsto & \frac{\sqrt{a^2 \cos(\theta)^2 + r^2}\sqrt{a^2+q^2-2\,mr+r^2}}{\sqrt{a^4+2\,a^2r^2+r^4-\left(a^4+a^2q^2-2\,a^2mr+a^2r^2\right)\sin(\theta)^2}} \end{array}$$

Electromagnetic field tensor

Let us first introduce the 1-form basis associated with Boyer-Lindquist coordinates:

In [14]:
```
dBL = BL.coframe() ; dBL
```

Out[14]: $(\mathcal{M}_0, (\mathrm{d}t, \mathrm{d}r, \mathrm{d}\theta, \mathrm{d}\phi))$

The electromagnetic field tensor F is formed as [cf. e.g. Eq. (33.5) of Misner, Thorne & Wheeler (1973)]

In [15]:
```
F = M.diff_form(2, name='F')
F.set_restriction( q/rho2^2 * (r^2-a^2*cos(th)^2)* dBL[1].wedge( dBL[0]
 - a*sin(th)^2* dBL[3] ) + \
    2*q/rho2^2 * a*r*cos(th)*sin(th)* dBL[2].wedge( (r^2+a^2)* dBL[3] -
 a* dBL[0] ) )
F.display()
```

Out[15]:
$$F = \left(\frac{a^2 q \cos(\theta)^2 - qr^2}{a^4 \cos(\theta)^4 + 2\,a^2r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}t \wedge \mathrm{d}r + \left(\frac{2\,a^2 qr \cos(\theta)\sin(\theta)}{a^4 \cos(\theta)^4 + 2\,a^2r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}t \wedge \mathrm{d}\theta + \left(\frac{\left(a^3 q \cos(\theta)^2 - aqr^2\right)\sin(\theta)^2}{a^4 \cos(\theta)^4 + 2\,a^2r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}r \wedge \mathrm{d}\phi + \left(\frac{2\left(a^3 qr + aqr^3\right)\cos(\theta)\sin(\theta)}{a^4 \cos(\theta)^4 + 2\,a^2r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}\theta \wedge \mathrm{d}\phi$$

The list of non-vanishing components:

In [16]:
```
F.display_comp()
```

Out[16]:

$$
\begin{aligned}
F_{\,t\,r} &= \frac{a^2 q \cos(\theta)^2 - q r^2}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4} \\
F_{\,t\,\theta} &= \frac{2\, a^2 q r \cos(\theta) \sin(\theta)}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4} \\
F_{\,r\,t} &= -\frac{a^2 q \cos(\theta)^2 - q r^2}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4} \\
F_{\,r\,\phi} &= \frac{\left(a^3 q \cos(\theta)^2 - a q r^2\right) \sin(\theta)^2}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4} \\
F_{\,\theta\,t} &= -\frac{2\, a^2 q r \cos(\theta) \sin(\theta)}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4} \\
F_{\,\theta\,\phi} &= \frac{2\, \left(a^3 q r + a q r^3\right) \cos(\theta) \sin(\theta)}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4} \\
F_{\,\phi\,r} &= -\frac{\left(a^3 q \cos(\theta)^2 - a q r^2\right) \sin(\theta)^2}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4} \\
F_{\,\phi\,\theta} &= -\frac{2\, \left(a^3 q r + a q r^3\right) \cos(\theta) \sin(\theta)}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4}
\end{aligned}
$$

The Hodge dual of F:

In [17]:
```
star_F = F.hodge_dual(g) ; star_F.display()
```

Out[17]:

$$
\begin{aligned}
\star F = & \left(\frac{2\, a q r \cos(\theta)}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4} \right) \mathrm{d}t \wedge \mathrm{d}r \\
& + \left(-\frac{\left(a^3 q \cos(\theta)^2 - a q r^2\right) \sin(\theta)}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4} \right) \mathrm{d}t \wedge \mathrm{d}\theta \\
& + \left(-\frac{2\, \left(a^4 q r \cos(\theta) \sin(\theta)^4 - \left(a^4 q r + a^2 q r^3\right) \cos(\theta) \sin(\theta)^2\right)}{a^6 \cos(\theta)^6 + 3\, a^4 r^2 \cos(\theta)^4 + 3\, a^2 r^4 \cos(\theta)^2 + r^6} \right) \mathrm{d}r \wedge \mathrm{d}\phi \\
& + \left(\frac{\left(a^4 q + a^2 q r^2\right) \sin(\theta)^3 - \left(a^4 q - q r^4\right) \sin(\theta)}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4} \right) \mathrm{d}\theta \wedge \mathrm{d}\phi
\end{aligned}
$$

Maxwell equations

Let us check that F obeys the two (source-free) Maxwell equations:

In [18]:
```
F.exterior_derivative().display()
```

Out[18]: $\mathrm{d}F = 0$

In [19]:
```
star_F.exterior_derivative().display()
```

Out[19]: $\mathrm{d} \star F = 0$

Levi-Civita Connection

The Levi-Civita connection ∇ associated with g:

In [20]:
```
nab = g.connection() ; print(nab)
```

```
Levi-Civita connection nabla_g associated with the Lorentzian metric g
on the 4-dimensional differentiable manifold M
```

Let us verify that the covariant derivative of g with respect to ∇ vanishes identically:

In [21]:
```
nab(g) == 0
```

Out[21]: True

Another view of the above property:

In [22]:
```
nab(g).display()
```

Out[22]: $\nabla_g g = 0$

The nonzero Christoffel symbols (skipping those that can be deduced by symmetry of the last two indices):

In [23]:
```
g.christoffel_symbols_display()
```

Out[23]:

$$\Gamma^{t}{}_{t\,r} = \frac{a^4m+a^2q^2r+q^2r^3-mr^4-\left(a^4m+a^2mr^2\right)\sin(\theta)^2}{2\,mr^5-r^6-\left(a^2+q^2\right)r^4-\left(a^6+a^4q^2-2\,a^4mr+a^4r^2\right)\cos(\theta)^4+2\,\left(2\,a^2mr^3-a^2r^4-\left(a^4+a^2q^2\right)r^2\right)\cos(\theta}$$

$$\Gamma^{t}{}_{t\,\theta} = \frac{\left(a^2q^2-2\,a^2mr\right)\cos(\theta)\sin(\theta)}{a^4\cos(\theta)^4+2\,a^2r^2\cos(\theta)^2+r^4}$$

$$\Gamma^{t}{}_{r\,\phi} = -\frac{a^3q^2r-a^3mr^2+2\,aq^2r^3-3\,amr^4-\left(a^5m+a^3q^2r-a^3mr^2\right)\cos(\theta)^4+\left(a^5m-2\,aq^2r^3+3\,amr^4\right)\cos(\theta}{2\,mr^5-r^6-\left(a^2+q^2\right)r^4-\left(a^6+a^4q^2-2\,a^4mr+a^4r^2\right)\cos(\theta)^4+2\,\left(2\,a^2mr^3-a^2r^4-\left(a^4+a^2q^2\right)r^2\right)\cos}$$

$$\Gamma^{t}{}_{\theta\,\phi} = \frac{\left(a^5q^2-2\,a^5mr\right)\cos(\theta)\sin(\theta)^5-\left(a^5q^2-2\,a^5mr+a^3q^2r^2-2\,a^3mr^3\right)\cos(\theta)\sin(\theta)^3}{a^6\cos(\theta)^6+3\,a^4r^2\cos(\theta)^4+3\,a^2r^4\cos(\theta)^2+r^6}$$

$$\Gamma^{r}{}_{t\,t} = \frac{mr^4-\left(2\,m^2+q^2\right)r^3+\left(a^2m+3\,mq^2\right)r^2-\left(a^4m+a^2mq^2-2\,a^2m^2r+a^2mr^2\right)\cos(\theta)^2-\left(a^2q^2+q^4\right)r}{a^6\cos(\theta)^6+3\,a^4r^2\cos(\theta)^4+3\,a^2r^4\cos(\theta)^2+r^6}$$

$$\Gamma^{r}{}_{t\,\phi} = -\frac{\left(amr^4-\left(2\,am^2+aq^2\right)r^3+\left(a^3m+3\,amq^2\right)r^2-\left(a^5m+a^3mq^2-2\,a^3m^2r+a^3mr^2\right)\cos(\theta)^2-\left(a^3q^2+aq^4\right.}{a^6\cos(\theta)^6+3\,a^4r^2\cos(\theta)^4+3\,a^2r^4\cos(\theta)^2+r^6}$$

$$\Gamma^{r}{}_{r\,r} = -\frac{a^2m+q^2r-mr^2-\left(a^2m-a^2r\right)\sin(\theta)^2}{2\,mr^3-r^4-\left(a^2+q^2\right)r^2-\left(a^4+a^2q^2-2\,a^2mr+a^2r^2\right)\cos(\theta)^2}$$

$$\Gamma^{r}{}_{r\,\theta} = -\frac{a^2\cos(\theta)\sin(\theta)}{a^2\cos(\theta)^2+r^2}$$

$$\Gamma^{r}{}_{\theta\,\theta} = \frac{2\,mr^2-r^3-\left(a^2+q^2\right)r}{a^2\cos(\theta)^2+r^2}$$

$$\Gamma^{r}{}_{\phi\,\phi} = \frac{\left(a^2mr^4-\left(2\,a^2m^2+a^2q^2\right)r^3+\left(a^4m+3\,a^2mq^2\right)r^2-\left(a^6m+a^4mq^2-2\,a^4m^2r+a^4mr^2\right)\cos(\theta)^2-\left(a^4q^2+a\right.\right. (\theta)^4+\left(2\,mr^6-r^7-\left(a^2+q^2\right)r^5+\left(2\,a^4mr^2-a^4r^3-\left(a^6+a^4q^2\right)r\right)\cos(\theta)^4+2\,\left(2\,a^2mr^4-a^2r^5-\left(a^4+a^2q^2\right.\right.\right. (\theta)^2}{a^6\cos(\theta)^6+3\,a^4r^2\cos(\theta)^4+3\,a^2r^4\cos(\theta)^2+r^6}$$

$$\Gamma^{\theta}{}_{t\,t} = \frac{\left(a^2q^2-2\,a^2mr\right)\cos(\theta)\sin(\theta)}{a^6\cos(\theta)^6+3\,a^4r^2\cos(\theta)^4+3\,a^2r^4\cos(\theta)^2+r^6}$$

$$\Gamma^{\theta}{}_{t\,\phi} = -\frac{\left(a^3q^2-2\,a^3mr+aq^2r^2-2\,amr^3\right)\cos(\theta)\sin(\theta)}{a^6\cos(\theta)^6+3\,a^4r^2\cos(\theta)^4+3\,a^2r^4\cos(\theta)^2+r^6}$$

$$\Gamma^{\theta}{}_{r\,r} = -\frac{a^2\cos(\theta)\sin(\theta)}{2\,mr^3-r^4-\left(a^2+q^2\right)r^2-\left(a^4+a^2q^2-2\,a^2mr+a^2r^2\right)\cos(\theta)^2}$$

$$\Gamma^{\theta}{}_{r\,\theta} = \frac{r}{a^2\cos(\theta)^2+r^2}$$

$$\Gamma^{\theta}{}_{\theta\,\theta} = -\frac{a^2\cos(\theta)\sin(\theta)}{a^2\cos(\theta)^2+r^2}$$

$$\Gamma^{\theta}{}_{\phi\,\phi} = -\frac{\left(\left(a^6+a^4q^2-2\,a^4mr+a^4r^2\right)\cos(\theta)^5-2\,\left(2\,a^2mr^3-a^2r^4-\left(a^4+a^2q^2\right)r^2\right)\cos(\theta)^3 \sin(\theta) -\left(a^4q^2-2\,a^4mr+2\,a^2q^2r^2-4\,a^2mr^3-a^2r^4-r^6\right)\cos(\theta)\right)}{a^6\cos(\theta)^6+3\,a^4r^2\cos(\theta)^4+3\,a^2r^4\cos(\theta)^2+r^6}$$

$$\Gamma^{\phi}{}_{t\,r} = \frac{a^3m\cos(\theta)^2+aq^2r-amr^2}{2\,mr^5-r^6-\left(a^2+q^2\right)r^4-\left(a^6+a^4q^2-2\,a^4mr+a^4r^2\right)\cos(\theta)^4+2\,\left(2\,a^2mr^3-a^2r^4-\left(a^4+a^2q^2\right)r^2\right)\cos(\theta}$$

$$\Gamma^{\phi}{}_{t\,\theta} = \frac{\left(aq^2-2\,amr\right)\cos(\theta)}{\left(a^4\cos(\theta)^4+2\,a^2r^2\cos(\theta)^2+r^4\right)\sin(\theta)}$$

$$\Gamma^{\phi}{}_{r\,\phi} = -\frac{a^2q^2r-a^2mr^2+q^2r^3-2\,mr^4+r^5-\left(a^4m-a^4r\right)\cos(\theta)^4+\left(a^4m-a^2mr^2+2\,a^2r^3\right)\cos(\theta)^2}{2\,mr^5-r^6-\left(a^2+q^2\right)r^4-\left(a^6+a^4q^2-2\,a^4mr+a^4r^2\right)\cos(\theta)^4+2\,\left(2\,a^2mr^3-a^2r^4-\left(a^4+a^2q^2\right)r^2\right)\cos}$$

$$\Gamma^{\phi}{}_{\theta\,\phi} = \frac{a^4\cos(\theta)\sin(\theta)^4-\left(2\,a^4+a^2q^2-2\,a^2mr+2\,a^2r^2\right)\cos(\theta)\sin(\theta)^2+\left(a^4+2\,a^2r^2+r^4\right)\cos(\theta)}{\left(a^4\cos(\theta)^4+2\,a^2r^2\cos(\theta)^2+r^4\right)\sin(\theta)}$$

Killing vectors

The default vector frame on the spacetime manifold is the coordinate basis associated with Boyer-Lindquist coordinates:

In [24]:
```
M.default_frame() is BL.frame()
```
Out[24]: True

In [25]:
```
BL.frame()
```
Out[25]: $\left(\mathcal{M}_0, \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\right)\right)$

Let us consider the first vector field of this frame:

In [26]:
```
xi = BL.frame()[0] ; xi
```
Out[26]: $\frac{\partial}{\partial t}$

In [27]:
```
print(xi)
```
```
Vector field d/dt on the Open subset M0 of the 4-dimensional differenti
able manifold M
```

The 1-form associated to it by metric duality is

In [28]:
```
xi_form = xi.down(g) ; xi_form.display()
```
Out[28]:

$$\left(-\frac{a^2 \cos(\theta)^2 + q^2 - 2\,mr + r^2}{a^2 \cos(\theta)^2 + r^2}\right) dt + \left(\frac{\left(aq^2 - 2\,amr\right) \sin(\theta)^2}{a^2 \cos(\theta)^2 + r^2}\right) d\phi$$

Its covariant derivative is

In [29]:
```
nab_xi = nab(xi_form) ; print(nab_xi) ; nab_xi.display()
```

```
Tensor field of type (0,2) on the Open subset M0 of the 4-dimensional d
ifferentiable manifold M
```

Out[29]:

$$\left(\frac{mr^4 - (2m^2+q^2)r^3 + (a^2m + 3mq^2)r^2 - (a^4m + a^2mq^2 - 2a^2m^2r + a^2mr^2)\cos(\theta)^2 - (a^2q^2+q^4)r}{2mr^5 - r^6 - (a^2+q^2)r^4 - (a^6 + a^4q^2 - 2a^4mr + a^4r^2)\cos(\theta)^4 + 2(2a^2mr^3 - a^2r^4 - (a^4+a^2q^2)r^2)\cos(\theta)^2}\right) dt \otimes dr + \left(-\frac{(a^2q^2 - 2a^2mr)\cos(\theta)\sin(\theta)}{a^4\cos(\theta)^4 + 2a^2r^2\cos(\theta)^2 + r^4}\right) dt \otimes d\theta$$
$$+ \left(-\frac{mr^4 - (2m^2+q^2)r^3 + (a^2m + 3mq^2)r^2 - (a^4m + a^2mq^2 - 2a^2m^2r + a^2mr^2)\cos(\theta)^2 - (a^2q^2+q^4)r}{2mr^5 - r^6 - (a^2+q^2)r^4 - (a^6 + a^4q^2 - 2a^4mr + a^4r^2)\cos(\theta)^4 + 2(2a^2mr^3 - a^2r^4 - (a^4+a^2q^2)r^2)\cos(\theta)^2}\right) dr \otimes dt + \left(-\frac{a^3m\cos(\theta)^4 - aq^2r + amr^2 - (a^3m - aq^2r + amr^2)\cos(\theta)^2}{a^4\cos(\theta)^4 + 2a^2r^2\cos(\theta)^2 + r^4}\right) dr \otimes d\phi + \left(\frac{(a^2q^2 - 2a^2mr)\cos(\theta)\sin(\theta)}{a^4\cos(\theta)^4 + 2a^2r^2\cos(\theta)^2 + r^4}\right) d\theta \otimes dt$$
$$+ \left(-\frac{(a^3q^2 - 2a^3mr + aq^2r^2 - 2amr^3)\cos(\theta)\sin(\theta)}{a^4\cos(\theta)^4 + 2a^2r^2\cos(\theta)^2 + r^4}\right) d\theta \otimes d\phi + \left(\frac{a^3m\sin(\theta)^4 - (a^3m + aq^2r - amr^2)\sin(\theta)^2}{a^4\cos(\theta)^4 + 2a^2r^2\cos(\theta)^2 + r^4}\right) d\phi \otimes dr + \left(\frac{(a^3q^2 - 2a^3mr + aq^2r^2 - 2amr^3)\cos(\theta)\sin(\theta)}{a^4\cos(\theta)^4 + 2a^2r^2\cos(\theta)^2 + r^4}\right) d\phi \otimes d\theta$$

Let us check that the vector field $\xi = \frac{\partial}{\partial t}$ obeys Killing equation:

In [30]:
```
nab_xi.symmetrize() == 0
```

Out[30]: True

Similarly, let us check that $\chi := \frac{\partial}{\partial \phi}$ is a Killing vector:

In [31]:
```
chi = BL.frame()[3] ; chi
```

Out[31]: $\frac{\partial}{\partial \phi}$

In [32]:
```
nab(chi.down(g)).symmetrize() == 0
```

Out[32]: True

Another way to check that ξ and χ are Killing vectors is the vanishing of the Lie derivative of the metric tensor along them:

In [33]:
```
g.lie_derivative(xi) == 0
```

Out[33]: True

In [34]:
```
g.lie_derivative(chi) == 0
```

Out[34]: True

Curvature

The Ricci tensor associated with g:

In [35]:
```
Ric = g.ricci() ; print(Ric)
```

```
Field of symmetric bilinear forms Ric(g) on the 4-dimensional different
iable manifold M
```

In [36]:
```
Ric.display()
```

Out[36]:

$$\mathrm{Ric}(g) = \left(-\frac{a^2q^2\cos(\theta)^2 - 2a^2q^2 - q^4 + 2mq^2r - q^2r^2}{a^6\cos(\theta)^6 + 3a^4r^2\cos(\theta)^4 + 3a^2r^4\cos(\theta)^2 + r^6}\right)\mathrm{d}t\otimes\mathrm{d}t$$
$$+\left(-\frac{2a^3q^2 + aq^4 - 2amq^2r + 2aq^2r^2 - \left(2a^3q^2 + aq^4 - 2amq^2r + 2aq^2r^2\right)\cos(\theta)^2}{a^6\cos(\theta)^6 + 3a^4r^2\cos(\theta)^4 + 3a^2r^4\cos(\theta)^2 + r^6}\right)$$
$$\otimes\mathrm{d}\phi + \left(\frac{q^2}{2mr^3 - r^4 - \left(a^2+q^2\right)r^2 - \left(a^4 + a^2q^2 - 2a^2mr + a^2r^2\right)\cos(\theta)^2}\right)\mathrm{d}r$$
$$\otimes\mathrm{d}r + \left(\frac{q^2}{a^2\cos(\theta)^2 + r^2}\right)\mathrm{d}\theta\otimes\mathrm{d}\theta$$
$$+\left(-\frac{2a^3q^2 + aq^4 - 2amq^2r + 2aq^2r^2 - \left(2a^3q^2 + aq^4 - 2amq^2r + 2aq^2r^2\right)\cos(\theta)^2}{a^6\cos(\theta)^6 + 3a^4r^2\cos(\theta)^4 + 3a^2r^4\cos(\theta)^2 + r^6}\right)\mathrm{d}$$
$$\otimes\mathrm{d}t + \left(-\frac{\left(a^6q^2 + a^4q^4 - 2a^4mq^2r + a^4q^2r^2\right)\sin(\theta)^6 - \left(a^4q^4 - 2a^4mq^2r + a^2q^4r^2 - 2a^2mq^2r^3\right)\sin(\theta)^4 - \left(a^6q^2 + 3a^4q^2r^2 + 3a^2q^2r^4 + q^2r^6\right)\sin(\theta)^2}{a^8\cos(\theta)^8 + 4a^6r^2\cos(\theta)^6 + 6a^4r^4\cos(\theta)^4 + 4a^2r^6\cos(\theta)^2 + r^8}\right)\mathrm{d}\phi$$
$$\otimes\mathrm{d}\phi$$

In [37]:
```
Ric[:]
```

Out[37]:

$$\left(\begin{array}{cc} -\frac{a^{2} q^{2} \cos\left(\theta\right)^{2} - 2 \, a^{2} q^{2} - q^{4} + 2 \, m q^{2} r - q^{2} r^{2}}{a^{6} \cos\left(\theta\right)^{6} + 3 \, a^{4} r^{2} \cos\left(\theta\right)^{4} + 3 \, a^{2} r^{4} \cos\left(\theta\right)^{2} + r^{6}} & \\ 0 & \frac{q^{2}}{2 \, m r^{3} - r^{4} - \left(a^{2} + q^{2}\right) r^{2} - \left(a^{4} + a^{2} q^{2} - \right.} \\ 0 & \\ -\frac{2 \, a^{3} q^{2} + a q^{4} - 2 \, a m q^{2} r + 2 \, a q^{2} r^{2} - \left(2 \, a^{3} q^{2} + a q^{4} - 2 \, a m q^{2} r + 2 \, a q^{2} r^{2}\right) \cos\left(\theta\right)^{2}}{a^{6} \cos\left(\theta\right)^{6} + 3 \, a^{4} r^{2} \cos\left(\theta\right)^{4} + 3 \, a^{2} r^{4} \cos\left(\theta\right)^{2} + r^{6}} & \end{array} \right.$$

Let us check that in the Kerr case, i.e. when $q = 0$, the Ricci tensor is zero:

In [38]:
```
Ric[:].subs(q=0)
```

Out[38]:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The Riemann curvature tensor associated with g:

In [39]:
```
R = g.riemann() ; print(R)
```

```
Tensor field Riem(g) of type (1,3) on the 4-dimensional differentiable
manifold M
```

The component $R^0{}_{101}$ of the Riemann tensor is

In [40]:
```
R[0,1,0,1]
```

Out[40]:

$$\frac{4 \, a^{2} q^{2} r^{2} - 3 \, a^{2} m r^{3} + 3 \, q^{2} r^{4} - 2 \, m r^{5} + \left(a^{4} q^{2} - 3 \, a^{4} m r\right) \cos\left(\theta\right)^{4} - \left(2 \, a^{4} q^{2} - 9 \, a^{4} m r + 2 \, a^{2} q^{2} r^{2} - 7 \, a^{2} m r^{3}\right) \cos\left(\theta\right)^{2}}{2 \, m r^{7} - r^{8} - \left(a^{2} + q^{2}\right) r^{6} - \left(a^{8} + a^{6} q^{2} - 2 \, a^{6} m r + a^{6} r^{2}\right) \cos\left(\theta\right)^{6} + 3 \left(2 \, a^{4} m r^{3} - a^{4} r^{4} - \left(a^{6} + a^{4} q^{2}\right) r^{2}\right) \cos\left(\theta\right)^{4} + 3 \left(2 \, a^{2} m r^{5} - a^{2} r^{6} - \left(a^{4} + a^{2} q^{2}\right) r^{4}\right) \cos\left(\theta\right)^{2}}$$

The expression in the uncharged limit (Kerr spacetime) is

In [41]:
```
R[0,1,0,1].expr().subs(q=0).simplify_rational()
```

Out[41]:

$$\frac{3 \, a^{4} m r \cos\left(\theta\right)^{4} + 3 \, a^{2} m r^{3} + 2 \, m r^{5} - \left(9 \, a^{4} m r + 7 \, a^{2} m r^{3}\right) \cos\left(\theta\right)^{2}}{a^{2} r^{6} - 2 \, m r^{7} + r^{8} + \left(a^{8} - 2 \, a^{6} m r + a^{6} r^{2}\right) \cos\left(\theta\right)^{6} + 3 \left(a^{6} r^{2} - 2 \, a^{4} m r^{3} + a^{4} r^{4}\right) \cos\left(\theta\right)^{4} + 3 \left(a^{4} r^{4} - 2 \, a^{2} m r^{5} + a^{2} r^{6}\right) \cos\left(\theta\right)^{2}}$$

while in the non-rotating limit (Reissner-Nordström spacetime), it is

In [42]:
```
R[0,1,0,1].expr().subs(a=0).simplify_rational()
```

Out[42]:

$$-\frac{3\,q^2 - 2\,mr}{q^2 r^2 - 2\,mr^3 + r^4}$$

In the Schwarzschild limit, it reduces to

In [43]:
```
R[0,1,0,1].expr().subs(a=0, q=0).simplify_rational()
```

Out[43]:

$$-\frac{2\,m}{2\,mr^2 - r^3}$$

Obviously, it vanishes in the flat space limit:

In [44]:
```
R[0,1,0,1].expr().subs(m=0, a=0, q=0)
```

Out[44]: 0

Bianchi identity

Let us check the Bianchi identity $\nabla_p R^i{}_{jkl} + \nabla_k R^i{}_{jlp} + \nabla_l R^i{}_{jpk} = 0$:

In [45]:
```
DR = nab(R) ; print(DR)  #long (takes a while)
```

```
Tensor field nabla_g(Riem(g)) of type (1,4) on the 4-dimensional differ
entiable manifold M
```

In [46]:
```
for i in M.irange():
    for j in M.irange():
        for k in M.irange():
            for l in M.irange():
                for p in M.irange():
                    print DR[i,j,k,l,p] + DR[i,j,l,p,k] + DR[i,j,p,k,l]
 ,
```

```
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
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0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
```

If the last sign in the Bianchi identity is changed to minus, the identity does no longer hold:

In [47]:
```
DR[0,1,2,3,1] + DR[0,1,3,1,2] + DR[0,1,1,2,3] # should be zero (Bianchi
 identity)
```

Out[47]: 0

In [48]:
```
DR[0,1,2,3,1] + DR[0,1,3,1,2] - DR[0,1,1,2,3] # note the change of the
second + to -
```

Out[48]:

$$-\frac{4\left(\left(a^5q^2 - 6\,a^5mr + a^3q^2r^2 - 6\,a^3mr^3\right)\cos\left(\theta\right)^3 \sin(\theta) - \left(5\,a^3q^2r^2 - 6\,a^3mr^3 + 5\,aq^2r^4 - 6\,amr^5\right)\cos(\theta)\right)}{2\,mr^7 - r^8 - \left(a^2 + q^2\right)r^6 - \left(a^8 + a^6q^2 - 2\,a^6mr + a^6r^2\right)\cos\left(\theta\right)^6 + 3\left(2\,a^4mr^3 - a^4r^4 - \left(a^6 + a^4q^2\right)r^2\right)\cos\left(\theta\right)^4 + 3\left(2\,a^2mr^5 - a^2r^6 - \left(a^4 + a^2q^2\right)r^4\right)\cos\left(\theta\right)^2}$$

Ricci scalar

The Ricci scalar $R = g^{ab}R_{ab}$ of the Kerr-Newman spacetime vanishes identically:

In [49]:
```
g.ricci_scalar().display()
```

Out[49]:

$$\begin{array}{llll} \mathrm{r}(g): & \mathcal{M} & \longrightarrow & \mathbb{R} \\ \text{on } \mathcal{M}_0: & (t,r,\theta,\phi) & \longmapsto & 0 \end{array}$$

Einstein equation

The Einstein tensor is

In [50]:
```
G = Ric - 1/2*g.ricci_scalar()*g ; print(G)
```

```
Field of symmetric bilinear forms +Ric(g) on the 4-dimensional differen
tiable manifold M
```

Since the Ricci scalar is zero, the Einstein tensor reduces to the Ricci tensor:

In [51]:
```
G == Ric
```

Out[51]: True

The invariant $F_{ab}F^{ab}$ of the electromagnetic field:

In [52]:
```
Fuu = F.up(g)
F2 = F['_ab']*Fuu['^ab'] ; print(F2)
```

```
Scalar field on the 4-dimensional differentiable manifold M
```

In [53]:
```
F2.display()
```

Out[53]:

$$\begin{array}{llll} & \mathcal{M} & \longrightarrow & \mathbb{R} \\ \text{on } \mathcal{M}_0: & (t,r,\theta,\phi) & \longmapsto & -\frac{2\left(a^4q^2\cos(\theta)^4-6\,a^2q^2r^2\cos(\theta)^2+q^2r^4\right)}{a^8\cos(\theta)^8+4\,a^6r^2\cos(\theta)^6+6\,a^4r^4\cos(\theta)^4+4\,a^2r^6\cos(\theta)^2+r^8} \end{array}$$

The energy-momentum tensor of the electromagnetic field:

In [54]:
```
Fud = F.up(g,0)
T = 1/(4*pi)*( F['_k.']*Fud['^k_.'] - 1/4*F2 * g ); print(T)
```

```
Tensor field of type (0,2) on the 4-dimensional differentiable manifold
 M
```

In [55]:
```
T[:]
```

Out[55]:

$$\left(\begin{array}{lll} -\frac{a^2q^2\cos(\theta)^2-2\,a^2q^2-q^4+2\,mq^2r-q^2r^2}{8\left(\pi a^6\cos(\theta)^6+3\,\pi a^4r^2\cos(\theta)^4+3\,\pi a^2r^4\cos(\theta)^2+\pi r^6\right)} & & \\ & 0 & \frac{q^2}{8\left(2\,\pi mr^3-\pi r^4-\left(\pi a^2+\pi q^2\right)r^2-\left(\pi a^4+\pi a^2q^2-2\,\pi a^2 r\right.\right.} \\ & 0 & \\ & & \\ -\frac{\left(2\,a^3q^2+aq^4-2\,amq^2r+2\,aq^2r^2\right)\sin(\theta)^2}{8\left(\pi a^6\cos(\theta)^6+3\,\pi a^4r^2\cos(\theta)^4+3\,\pi a^2r^4\cos(\theta)^2+\pi r^6\right)} & & \end{array}\right.$$

Check of the Einstein equation:

In [56]:
```
G == 8*pi*T
```
Out[56]: True

Kretschmann scalar

The tensor $R^\flat$, of components $R_{abcd} = g_{am} R^m{}_{bcd}$:

In [57]:
```
dR = R.down(g) ; print(dR)
```
```
Tensor field of type (0,4) on the 4-dimensional differentiable manifold
 M
```

The tensor $R^\sharp$, of components $R^{abcd} = g^{bp} g^{cq} g^{dr} R^a{}_{pqr}$:

In [58]:
```
uR = R.up(g) ; print(uR)
```
```
Tensor field of type (4,0) on the 4-dimensional differentiable manifold
 M
```

The Kretschmann scalar $K := R^{abcd} R_{abcd}$:

In [59]:
```
Kr_scalar = uR['^ijkl']*dR['_ijkl']
Kr_scalar.display()
```
Out[59]:

$$\mathcal{M} \longrightarrow \mathbb{R}$$

$$8 \Big(6\, m^2 r^8 - 12\, (m^3 + m q^2) r^7 + (6\, a^2 m^2 + 30\, m^2 q^2 + 7\, q^4) r^6 - 6\, (a^8 m^2 + a^6 m^2 q^2 \ldots$$
$$(6\, a^2 m q^2 + 13\, m q^4) r^5 + 7\, (a^2 q^4 + q^6) r^4$$
$$+ (7\, a^6 q^4 + 7\, a^4 q^6 + 90\, a^4 m^2 r^4 - 60\, (3\, a^4 m^3 + a^4 m q^2) r^3 + (90\, a^6 m^2 + 210\, a^4 \ldots$$
$$(30\, a^6 m q^2 + 37\, a^4 m q^4) r)$$
$$(45\, a^2 m^2 r^6 - 30\, (3\, a^2 m^3 + 2\, a^2 m q^2) r^5 + (45\, a^4 m^2 + 165\, a^2 m^2 q^2 + 17\, a^2 q^4)$$
$$(a^4 q^4 + a^2 q^6) r^2)$$
$$(\theta)^2 \Big)$$

$$\text{on } \mathcal{M}_0 : \quad (t, r, \theta, \phi) \longmapsto -\frac{\ldots}{2\, m r^{13} - r^{14} - (a^2 + q^2) r^{12} - (a^{14} + a^{12} q^2 - 2\, a^{12} m r + a^{12} r^2) \cos(\theta)^{12} + 6\, (2 \ldots}$$
$$(\theta)^{10} + 15\, (2\, a^8 m r^5 - a^8 r^6 - (a^{10} + a^8 q^2) r^4) \cos(\theta)^8 + 20\, (2\, a^6 m r^7 - a^6 r^8 \ldots$$
$$(2\, a^4 m r^9 - a^4 r^{10} - (a^6 + a^4 q^2) r^8) \cos(\theta)^4 + 6\, (2\, a^2 m r^{11} - a^2 r^{12} - (a^4 + a^2 \ldots$$

A variant of this expression can be obtained by invoking the `factor()` method on the coordinate function representing the scalar field in the manifold's default chart:

In [60]:
```
Kr = Kr_scalar.coord_function()
Kr.factor()
```
Out[60]:

$$-\frac{8 \left(6\, a^6 m^2 \cos(\theta)^6 - 7\, a^4 q^4 \cos(\theta)^4 + 60\, a^4 m q^2 r \cos(\theta)^4 - 90\, a^4 m^2 r^2 \cos(\theta)^4 + 34\, a^2 q^4 r^2 \cos(\theta)^2 - 120\, a^2 m q^2 r^3 \cos(\theta)^2 + 90\, a^2 m^2 r^4 \cos(\theta)^2 - 7\, q^4 r^4 + 12\, m q^2 r^5 - 6\, m^2 r^6\right)}{\left(a^2 \cos(\theta)^2 + r^2\right)^6}$$

As a check, we can compare Kr to the formula given by R. Conn Henry, Astrophys. J. **535**, 350 (2000):

In [61]:
```
Kr == 8/(r^2+(a*cos(th))^2)^6 *(
          6*m^2*(r^6 - 15*r^4*(a*cos(th))^2 + 15*r^2*(a*cos(th))^4 - (a
*cos(th))^6)
        - 12*m*q^2*r*(r^4 - 10*(a*r*cos(th))^2 + 5*(a*cos(th))^4)
        + q^4*(7*r^4 - 34*(a*r*cos(th))^2 + 7*(a*cos(th))^4) )
```

Out[61]: True

The Schwarzschild value of the Kretschmann scalar is recovered by setting $a = 0$ and $q = 0$:

In [62]:
```
Kr.expr().subs(a=0, q=0)
```

Out[62]: $\frac{48\,m^2}{r^6}$

Let us plot the Kretschmann scalar for $m = 1$, $a = 0.9$ and $q = 0.5$:

In [63]:
```
K1 = Kr.expr().subs(m=1, a=0.9, q=0.5)
plot3d(K1, (r,1,3), (th, 0, pi), viewer=viewer3D, axes_labels=['r', 'th
eta', 'Kr'])
```

Out[63]: