3+1 Simon-Mars tensor in Kerr spacetime

This worksheet demonstrates a few capabilities of SageManifolds (version 1.0, as included in SageMath 7.5) in computations regarding 3+1 slicing of Kerr spacetime. In particular, it implements the computation of the 3+1 decomposition of the Simon-Mars tensor as given in the article arXiv:1412.6542.

Click here to download the worksheet file (ipynb format). To run it, you must start SageMath with the Jupyter notebook, via the command `sage -n jupyter`

NB: a version of SageMath at least equal to 7.5 is required to run this worksheet:

In [1]:
```
version()
```

Out[1]:
```
'SageMath version 7.5.1, Release Date: 2017-01-15'
```

First we set up the notebook to display mathematical objects using LaTeX rendering:

In [2]:
```
%display latex
```

Since some computations are quite long, we ask for running them in parallel on 8 cores:

In [3]:
```
Parallelism().set(nproc=8)
```

Spacelike hypersurface

We consider some hypersurface Σ of a spacelike foliation $(\Sigma_t)_{t\in\mathbb{R}}$ of Kerr spacetime; we declare Σ_t as a 3-dimensional manifold:

In [4]:
```
Sig = Manifold(3, 'Sigma', r'\Sigma', start_index=1)
```

The two Kerr parameters:

In [5]:
```
var('m, a')
assume(m>0)
assume(a>0)
```

Riemannian metric on Σ

The variables introduced so far satisfy the following assumptions:

Without any loss of generality (for $m \neq 0$), we may set $m = 1$:

In [6]:
```
m=1
assume(a<1)
```

In [7]:
```
#a=1 # extreme Kerr
```

On the hypersurface Σ, we are using coordinates (r, y, ϕ) that are related to the standard Boyer-Lindquist coordinates (r, θ, ϕ) by $y = \cos\theta$:

In [8]:
```
X.<r,y,ph> = Sig.chart(r'r:(1+sqrt(1-a^2),+oo) y:(-1,1) ph:(0,2*pi):\ph
i')
print(X) ; X
```

```
Chart (Sigma, (r, y, ph))
```

Out[8]: $(\Sigma, (r, y, \phi))$

Riemannian metric on Σ

The variables introduced so far obey the following assumptions:

In [9]:
```
assumptions()
```

Out[9]:

$$[m > 0, a > 0, a < 1, \mathtt{r\ is\ real}, \mathtt{y\ is\ real}, y > (-1), y < 1, \mathtt{ph\ is\ real}, \phi > 0, \phi < 2\pi]$$

Some shortcut notations:

In [10]:
```
rho2 = r^2 + a^2*y^2
Del = r^2 -2*m*r + a^2
AA2 = rho2*(r^2 + a^2) + 2*a^2*m*r*(1-y^2)
BB2 = r^2 + a^2 + 2*a^2*m*r*(1-y^2)/rho2
```

The metric h induced by the spacetime metric g on Σ:

In [11]:
```
gam = Sig.riemannian_metric('gam', latex_name=r'\gamma')
gam[1,1] = rho2/Del
gam[2,2] = rho2/(1-y^2)
gam[3,3] = BB2*(1-y^2)
gam.display()
```

Out[11]:

$$\gamma = \left(\frac{a^2 y^2 + r^2}{a^2 + r^2 - 2\, r}\right) \mathrm{d}r \otimes \mathrm{d}r + \left(-\frac{a^2 y^2 + r^2}{y^2 - 1}\right) \mathrm{d}y \otimes \mathrm{d}y + \left(\frac{2\,(y^2 - 1) a^2 r}{a^2 y^2 + r^2} - a^2 - r^2\right)(y^2 - 1) \mathrm{d}\phi \otimes \mathrm{d}\phi$$

A matrix view of the components w.r.t. coordinates (r, y, ϕ):

In [12]:
```
gam[:]
```

Out[12]:

$$\begin{pmatrix} \frac{a^2 y^2 + r^2}{a^2 + r^2 - 2\, r} & 0 & 0 \\ 0 & -\frac{a^2 y^2 + r^2}{y^2 - 1} & 0 \\ 0 & 0 & \left(\frac{2\,(y^2 - 1) a^2 r}{a^2 y^2 + r^2} - a^2 - r^2\right)(y^2 - 1) \end{pmatrix}$$

Lapse function and shift vector

In [13]:
```
N = Sig.scalar_field(sqrt(Del / BB2), name='N')
print(N)
N.display()
```

```
Scalar field N on the 3-dimensional differentiable manifold Sigma
```

Out[13]:

$$\begin{array}{llcl} N: & \Sigma & \longrightarrow & \mathbb{R} \\ & (r, y, \phi) & \longmapsto & \sqrt{-\frac{a^2+r^2-2\,r}{\frac{2\,(y^2-1)a^2r}{a^2y^2+r^2}-a^2-r^2}} \end{array}$$

In [14]:
```
b = Sig.vector_field('beta', latex_name=r'\beta')
b[3] = -2*m*r*a/AA2
# unset components are zero
b.display()
```

Out[14]:

$$\beta = \left(\frac{2\,ar}{2\,(y^2-1)a^2r - (a^2y^2+r^2)(a^2+r^2)}\right)\frac{\partial}{\partial\phi}$$

Extrinsic curvature of Σ

We use the formula

$$K_{ij} = \frac{1}{2N}\mathcal{L}_\beta \gamma_{ij}$$

which is valid for any stationary spacetime:

In [15]:
```
K = gam.lie_der(b) / (2*N)
K.set_name('K')
print(K) ; K.display()
```

```
Field of symmetric bilinear forms K on the 3-dimensional differentiable
 manifold Sigma
```

Out[15]:

$$\left(\frac{(a^3r^2+3\,ar^4+(a^5-a^3r^2)y^4-(a^5+3\,ar^4)y^2)\sqrt{a^2r^2+r^4+2\,a^2r+(a^4 \cdots}}{(a^2r^4+r^6+2\,a^2r^3+(a^6+a^4r^2-2\,a^4r)y^4+2\,(a^4r^2+a^2r^4+a^4r-a^2r^3)y^2\sqrt{a \cdots}} \right)$$

Check (comparison with known formulas):

In [16]:
```
Krp = a*m*(1-y^2)*(3*r^4+a^2*r^2+a^2*(r^2-a^2)*y^2) / rho2^2/sqrt(Del*B
B2)
Krp
```

Out[16]:

$$\frac{((a^2-r^2)a^2y^2-a^2r^2-3\,r^4)(y^2-1)a}{(a^2y^2+r^2)^2\sqrt{-\left(\frac{2\,(y^2-1)a^2r}{a^2y^2+r^2}-a^2-r^2\right)(a^2+r^2-2\,r)}}$$

In [17]:
```
K[1,3] - Krp
```

Out[17]: 0

In [18]:
```
Kyp = 2*m*r*a^3*(1-y^2)*y*sqrt(Del)/rho2^2/sqrt(BB2)
Kyp
```

Out[18]:

$$-\frac{2\sqrt{a^2+r^2-2r}\left(y^2-1\right)a^3ry}{\left(a^2y^2+r^2\right)^2\sqrt{-\frac{2\left(y^2-1\right)a^2r}{a^2y^2+r^2}+a^2+r^2}}$$

In [19]:
```
K[2,3] - Kyp
```

Out[19]: 0

For now on, we use the expressions Krp and Kyp above for $K_{r\phi}$ and K_{ry}, respectively:

In [20]:
```
K1 = Sig.sym_bilin_form_field('K')
K1[1,3] = Krp
K1[2,3] = Kyp
K = K1
K.display()
```

Out[20]:

$$K = \left(\frac{\left(\left(a^2-r^2\right)a^2y^2-a^2r^2-3r^4\right)\left(y^2-1\right)a}{\left(a^2y^2+r^2\right)^2\sqrt{-\left(\frac{2\left(y^2-1\right)a^2r}{a^2y^2+r^2}-a^2-r^2\right)\left(a^2+r^2-2r\right)}}\right)\mathrm{d}r\otimes\mathrm{d}\phi$$

$$+\left(-\frac{2\sqrt{a^2+r^2-2r}\left(y^2-1\right)a^3ry}{\left(a^2y^2+r^2\right)^2\sqrt{-\frac{2\left(y^2-1\right)a^2r}{a^2y^2+r^2}+a^2+r^2}}\right)\mathrm{d}y\otimes\mathrm{d}\phi$$

$$+\left(\frac{\left(\left(a^2-r^2\right)a^2y^2-a^2r^2-3r^4\right)\left(y^2-1\right)a}{\left(a^2y^2+r^2\right)^2\sqrt{-\left(\frac{2\left(y^2-1\right)a^2r}{a^2y^2+r^2}-a^2-r^2\right)\left(a^2+r^2-2r\right)}}\right)\mathrm{d}\phi\otimes\mathrm{d}r$$

$$+\left(-\frac{2\sqrt{a^2+r^2-2r}\left(y^2-1\right)a^3ry}{\left(a^2y^2+r^2\right)^2\sqrt{-\frac{2\left(y^2-1\right)a^2r}{a^2y^2+r^2}+a^2+r^2}}\right)\mathrm{d}\phi\otimes\mathrm{d}y$$

The type-(1,1) tensor $K^\sharp$ of components $K^i{}_j = \gamma^{ik}K_{kj}$:

In [21]:
```
Ku = K.up(gam, 0)
print(Ku) ; Ku.display()
```

```
Tensor field of type (1,1) on the 3-dimensional differentiable manifold
 Sigma
```

Out[21]:

$$\left(\frac{\left(a^3 r^2 + 3\, a r^4 - \left(a^5 - a^3 r^2\right) y^2\right) \sqrt{a}}{\left(a^2 r^4 + r^6 + 2\, a^2 r^3 + \left(a^6 + a^4 r^2 - 2\, a^4 r\right) y^4 + 2\, \left(a^4 r^2 + a^2 r^4 + a^4 r - a^2 r^3\right) y^2 \sqrt{a}\right.} \right)$$

We may check that the hypersurface Σ is maximal, i.e. that $K^k_{\ \ k} = 0$:

In [22]:
```
trK = Ku.trace()
print(trK)
```

```
Scalar field zero on the 3-dimensional differentiable manifold Sigma
```

Connection and curvature

Let us call D the Levi-Civita connection associated with γ:

In [23]:
```
D = gam.connection(name='D')
print(D) ; D
```

```
Levi-Civita connection D associated with the Riemannian metric gam on t
he 3-dimensional differentiable manifold Sigma
```

Out[23]: D

The Ricci tensor associated with γ:

In [24]:
```
Ric = gam.ricci()
print(Ric) ; Ric
```

```
Field of symmetric bilinear forms Ric(gam) on the 3-dimensional differe
ntiable manifold Sigma
```

Out[24]: $\mathrm{Ric}\left(\gamma\right)$

In [25]:
```
Ric[1,1]
```

Out[25]:

$$-\frac{8a^4r^7+7a^2r^9+2r^{11}+5a^6r^4+2a^4r^6-7a^2r^8+\left(3a^{10}r+3a^6r^5+a^{10}-14a^8r^2-11a^6r^4+6\left(a^8+2a^6\right)r^3\right)y^6+\left(3a^6-4a^4\right)r^5-\left(9a^{10}r+4a^4r^7+a^{10}-30a^8r^2-35a^6r^4-16a^4r^6+\left(17a^6+4a^4\right)r^5+2\left(11a^8+12a^6\right)r^3\right)y^4-\left(16a^4r^7+5a^2r^9+16a^8r^2+29a^6r^4+18a^4r^6-7a^2r^8+\left(17a^6-8a^4\right)r^5+6\left(a^8-2a^6\right)r^3\right)y^2}{3a^2r^{12}+r^{14}+6a^4r^9-2r^{13}+4a^6r^6+\left(3a^4-8a^2\right)r^{10}+\left(a^6-4a^4\right)r^8+\left(a^{14}+a^8r^6-6a^{12}r-6a^8r^5+3\left(a^{10}+4a^8\right)r^4-4\left(3a^{10}+2a^8\right)r^3+3\left(a^{12}+4a^{10}\right)r^2\right)y^8+4\left(a^6-2a^4\right)r^7+4\left(a^6r^8+a^{12}r-5a^6r^7+\left(3a^8+8a^6\right)r^6-\left(9a^8+4a^6\right)r^5+\left(3a^{10}+4a^8\right)r^4-\left(3a^{10}-4a^8\right)r^3+\left(a^{12}-4a^{10}\right)r^2\right)y^6+2\left(3a^4r^{10}-12a^4r^9+2a^{10}r^2+16a^6r^5+\left(9a^6+14a^4\right)r^8-2\left(9a^6+2a^4\right)r^7+3\left(3a^8-2a^6\right)r^6+3\left(a^{10}-6a^8\right)r^4+2\left(3a^{10}-2a^8\right)r^3\right)y^4+4\left(a^2r^{12}-3a^4r^9-3a^2r^{11}+2a^8r^4+\left(3a^4+2a^2\right)r^{10}+3\left(a^6-2a^4\right)r^8+\left(3a^6+4a^4\right)r^7+\left(a^8-6a^6\right)r^6+\left(3a^8-4a^6\right)r^5\right)y^2}$$

In [26]:
```
Ric[1,2]
```

Out[26]:

$$\frac{\left(3a^{10}+6a^8r^2+3a^6r^4-4a^8r-8a^6r^3\right)y^5-2\left(3a^8r^2+6a^6r^4+3a^4r^6-2a^8r-12a^6r^3-6a^4r^5\right)y^3-\left(9a^6r^4+18a^4r^6+9a^2r^8+16a^6r^3+12a^4r^5\right)y}{a^4r^8+2a^2r^{10}+r^{12}+4a^4r^7+4a^2r^9+4a^4r^6+\left(a^{12}+a^8r^4-4a^{10}r-4a^8r^3+2\left(a^{10}+2a^8\right)r^2\right)y^8+4\left(a^6r^6+a^{10}r-2a^8r^3-3a^6r^5+2\left(a^8+a^6\right)r^4+\left(a^{10}-2a^8\right)r^2\right)y^6+2\left(3a^4r^8+6a^8r^3-6a^4r^7+2a^8r^2+2\left(3a^6+a^4\right)r^6+\left(3a^8-8a^6\right)r^4\right)y^4+4\left(2a^4r^8+a^2r^{10}+3a^6r^5+2a^4r^7-a^2r^9+2a^6r^4+\left(a^6-2a^4\right)r^6\right)y^2}$$

In [27]:
```
Ric[1,3]
```

Out[27]: 0

In [28]:
```
Ric[2,2]
```

Out[28]:

$$\frac{\begin{array}{l} 7\,a^{4} r^{7} + 5\,a^{2} r^{9} + r^{11} + 6\,a^{6} r^{4} + 4\,a^{4} r^{6} - 2\,a^{2} r^{8} + 2 \\ \left(3\,a^{10} r + 3\,a^{6} r^{5} - 10\,a^{8} r^{2} - 10\,a^{6} r^{4} + 2\,\left(3\,a^{8} + 4\,a^{6}\right) r^{3}\right) y^{6} + \left(3\,a^{6} - 8\,a^{4}\right) r^{5} \\ - \left(9\,a^{10} r - a^{4} r^{7} - 34\,a^{8} r^{2} - 36\,a^{6} r^{4} - 2\,a^{4} r^{6} + \left(7\,a^{6} + 8\,a^{4}\right) r^{5} y^{4} - 2 \right. \\ \quad \left. + \left(17\,a^{8} + 32\,a^{6}\right) r^{3}\right) \\ \left(7\,a^{4} r^{7} + 2\,a^{2} r^{9} + 7\,a^{8} r^{2} + 11\,a^{6} r^{4} + 3\,a^{4} r^{6} - a^{2} r^{8} + 8\,\left(a^{6} - a^{4}\right) r^{5} y^{2} \right. \\ \left. + \left(3\,a^{8} - 8\,a^{6}\right) r^{3}\right) \end{array}}{\begin{array}{l} a^{4} r^{8} + 2\,a^{2} r^{10} + r^{12} + 4\,a^{4} r^{7} + 4\,a^{2} r^{9} \\ - \left(a^{12} + a^{8} r^{4} - 4\,a^{10} r - 4\,a^{8} r^{3} + 2\,\left(a^{10} + 2\,a^{8}\right) r^{2}\right) y^{10} + 4\,a^{4} r^{6} \\ + \left(a^{12} - 4\,a^{6} r^{6} - 8\,a^{10} r + 4\,a^{8} r^{3} + 12\,a^{6} r^{5} - \left(7\,a^{8} + 8\,a^{6}\right) r^{4} - 2 y^{8} - 2 \right. \\ \quad \left. \left(a^{10} - 6\,a^{8}\right) r^{2}\right) \\ \left(3\,a^{4} r^{8} - 2\,a^{10} r + 10\,a^{8} r^{3} + 6\,a^{6} r^{5} - 6\,a^{4} r^{7} + 2\,\left(2\,a^{6} + a^{4}\right) r^{6} y^{6} - 2 \right. \\ \left. - \left(a^{8} + 12\,a^{6}\right) r^{4} - 2\,\left(a^{10} - 3\,a^{8}\right) r^{2}\right) \\ \left(a^{4} r^{8} + 2\,a^{2} r^{10} - 6\,a^{8} r^{3} + 6\,a^{6} r^{5} + 10\,a^{4} r^{7} - 2\,a^{2} r^{9} - 2\,a^{8} r^{2} - 2 y^{4} \right. \\ \left. \left(2\,a^{6} + 3\,a^{4}\right) r^{6} - 3\,\left(a^{8} - 4\,a^{6}\right) r^{4}\right) \\ + \left(7\,a^{4} r^{8} + 2\,a^{2} r^{10} - r^{12} + 12\,a^{6} r^{5} + 4\,a^{4} r^{7} - 8\,a^{2} r^{9} + 8\,a^{6} r^{4} + 4 y^{2} \right. \\ \quad \left. \left(a^{6} - 3\,a^{4}\right) r^{6}\right) \end{array}}$$

In [29]:
```
Ric[2,3]
```

Out[29]: 0

In [30]:
```
Ric[3,3]
```

Out[30]:

$$\frac{\begin{array}{l} a^{4} r^{7} + 2\,a^{2} r^{9} + r^{11} + a^{6} r^{4} + 10\,a^{4} r^{6} + 13\,a^{2} r^{8} + 4\,a^{4} r^{5} \\ + \left(3\,a^{10} r + 3\,a^{6} r^{5} + a^{10} - 18\,a^{8} r^{2} - 15\,a^{6} r^{4} + 2\,\left(3\,a^{8} + 10\,a^{6}\right) r^{3}\right) y^{8} \\ - \left(3\,a^{10} r - 5\,a^{4} r^{7} + 2\,a^{10} - 38\,a^{8} r^{2} - 22\,a^{6} r^{4} + 2\,a^{4} r^{6} - \left(7\,a^{6} - 4\,a^{4}\right) r^{5} y^{6} \right. \\ \quad \left. + \left(a^{8} + 60\,a^{6}\right) r^{3}\right) \\ - \left(3\,a^{4} r^{7} - a^{2} r^{9} - a^{10} + 22\,a^{8} r^{2} - 2\,a^{6} r^{4} - 14\,a^{4} r^{6} - 13\,a^{2} r^{8} + 3 y^{4} \right. \\ \quad \left. \left(3\,a^{6} - 4\,a^{4}\right) r^{5} + 5\,\left(a^{8} - 12\,a^{6}\right) r^{3}\right) \\ - \left(3\,a^{4} r^{7} + 3\,a^{2} r^{9} + r^{11} - 2\,a^{8} r^{2} + 10\,a^{6} r^{4} + 22\,a^{4} r^{6} + 26\,a^{2} r^{8} + 20\,a^{6} r^{3} y^{2} \right. \\ \quad \left. + \left(a^{6} + 12\,a^{4}\right) r^{5}\right) \end{array}}{\begin{array}{l} a^{2} r^{10} + r^{12} + 2\,a^{2} r^{9} + \left(a^{12} + a^{10} r^{2} - 2\,a^{10} r\right) y^{10} \\ + \left(5\,a^{10} r^{2} + 5\,a^{8} r^{4} + 2\,a^{10} r - 8\,a^{8} r^{3}\right) y^{8} + 2 \\ \left(5\,a^{8} r^{4} + 5\,a^{6} r^{6} + 4\,a^{8} r^{3} - 6\,a^{6} r^{5}\right) y^{6} + 2 \\ \left(5\,a^{6} r^{6} + 5\,a^{4} r^{8} + 6\,a^{6} r^{5} - 4\,a^{4} r^{7}\right) y^{4} \\ + \left(5\,a^{4} r^{8} + 5\,a^{2} r^{10} + 8\,a^{4} r^{7} - 2\,a^{2} r^{9}\right) y^{2} \end{array}}$$

The scalar curvature $R = \gamma^{ij} R_{ij}$:

In [31]:
```
R = gam.ricci_scalar(name='R')
print(R)
R.display()
```

```
Scalar field R on the 3-dimensional differentiable manifold Sigma
```

Out[31]:

$$\begin{array}{llcl} \mathrm{r}(\gamma): & \Sigma & \longrightarrow & \mathbb{R} \\ & (r, y, \phi) & \longmapsto & \dfrac{\begin{array}{l} 2\,\left(a^{6} r^{4}+6\,a^{4} r^{6}+9\,a^{2} r^{8}-\left(a^{10}-6\,a^{8} r^{2}-3\,a^{6} r^{4}+8\,a^{6} r^{3}\right) y^{6}+\left(a^{10}-8\,a^{8} r^{2}-3\,a^{6} r^{4}-6\,\cdots\right.\right. \\ \left.+\left(2\,a^{8} r^{2}-a^{6} r^{4}-9\,a^{2} r^{8}-8\,a^{6} r^{3}\right) y^{2}\right) \end{array}}{\begin{array}{l} a^{4} r^{10}+2\,a^{2} r^{12}+r^{14}+4\,a^{4} r^{9}+4\,a^{2} r^{11}+4\,a^{4} r^{8}+\left(a^{14}+a^{10} r^{4}-4\,a^{12} r-4\,a^{10} r^{3}+2\,\left(a^{1\cdots}\right.\right. \\ +\left(5\,a^{8} r^{6}+4\,a^{12} r-12\,a^{10} r^{3}-16\,a^{8} r^{5}+2\,\left(5\,a^{10}+6\,a^{8}\right) r^{4}+\left(5\,a^{12}-8\,a^{10}\right) r^{2}\right) y^{8}+2 \\ \left(5\,a^{6} r^{8}+8\,a^{10} r^{3}-4\,a^{8} r^{5}-12\,a^{6} r^{7}+2\,a^{10} r^{2}+2\,\left(5\,a^{8}+3\,a^{6}\right) r^{6}+\left(5\,a^{10}-12\,a^{8}\right) r^{4}\right)\cdots \\ \left(5\,a^{4} r^{10}+12\,a^{8} r^{5}+4\,a^{6} r^{7}-8\,a^{4} r^{9}+6\,a^{8} r^{4}+2\,\left(5\,a^{6}+a^{4}\right) r^{8}+\left(5\,a^{8}-12\,a^{6}\right) r^{6}\right) y^{4} \\ +\left(10\,a^{4} r^{10}+5\,a^{2} r^{12}+16\,a^{6} r^{7}+12\,a^{4} r^{9}-4\,a^{2} r^{11}+12\,a^{6} r^{6}+\left(5\,a^{6}-8\,a^{4}\right) r^{8}\right) y^{2} \end{array}} \end{array}$$

Test: 3+1 Einstein equations

Let us check that the vacuum 3+1 Einstein equations are satisfied.

We start by the contraint equations:

Hamiltonian constraint

Let us first evaluate the term $K_{ij} K^{ij}$:

In [32]:
```
Kuu = Ku.up(gam, 1)
trKK = K['_ij']*Kuu['^ij']
print(trKK) ; trKK.display()
```

```
Scalar field on the 3-dimensional differentiable manifold Sigma
```

Out[32]:

$$\begin{array}{lcl} \Sigma & \longrightarrow & \mathbb{R} \\ (r, y, \phi) & \longmapsto & \dfrac{\begin{array}{l} 2\,\left(a^{6} r^{4}+6\,a^{4} r^{6}+9\,a^{2} r^{8}-\left(a^{10}-6\,a^{8} r^{2}-3\,a^{6} r^{4}+8\,a^{6} r^{3}\right) y^{6}+\left(a^{10}-8\,a^{8} r^{2}-3\,a^{6} r^{4}-6\,a^{4} r^{6}+1\cdots\right.\right. \\ \left.+\left(2\,a^{8} r^{2}-a^{6} r^{4}-9\,a^{2} r^{8}-8\,a^{6} r^{3}\right) y^{2}\right) \end{array}}{\begin{array}{l} a^{4} r^{10}+2\,a^{2} r^{12}+r^{14}+4\,a^{4} r^{9}+4\,a^{2} r^{11}+4\,a^{4} r^{8}+\left(a^{14}+a^{10} r^{4}-4\,a^{12} r-4\,a^{10} r^{3}+2\,\left(a^{12}+2\,a^{10}\cdots\right.\right. \\ +\left(5\,a^{8} r^{6}+4\,a^{12} r-12\,a^{10} r^{3}-16\,a^{8} r^{5}+2\,\left(5\,a^{10}+6\,a^{8}\right) r^{4}+\left(5\,a^{12}-8\,a^{10}\right) r^{2}\right) y^{8}+2 \\ \left(5\,a^{6} r^{8}+8\,a^{10} r^{3}-4\,a^{8} r^{5}-12\,a^{6} r^{7}+2\,a^{10} r^{2}+2\,\left(5\,a^{8}+3\,a^{6}\right) r^{6}+\left(5\,a^{10}-12\,a^{8}\right) r^{4}\right) y^{6}+2 \\ \left(5\,a^{4} r^{10}+12\,a^{8} r^{5}+4\,a^{6} r^{7}-8\,a^{4} r^{9}+6\,a^{8} r^{4}+2\,\left(5\,a^{6}+a^{4}\right) r^{8}+\left(5\,a^{8}-12\,a^{6}\right) r^{6}\right) y^{4} \\ +\left(10\,a^{4} r^{10}+5\,a^{2} r^{12}+16\,a^{6} r^{7}+12\,a^{4} r^{9}-4\,a^{2} r^{11}+12\,a^{6} r^{6}+\left(5\,a^{6}-8\,a^{4}\right) r^{8}\right) y^{2} \end{array}} \end{array}$$

The vacuum Hamiltonian constraint equation is

$$R + K^2 - K_{ij} K^{ij} = 0$$

In [33]:
```
Ham = R + trK^2 - trKK
print(Ham) ; Ham.display()
```

```
Scalar field zero on the 3-dimensional differentiable manifold Sigma
```

Out[33]:

$$\begin{array}{llcl} 0: & \Sigma & \longrightarrow & \mathbb{R} \\ & (r, y, \phi) & \longmapsto & 0 \end{array}$$

Momentum constraint

In vaccum, the momentum constraint is

$$D_j K^j_{\ \ i} - D_i K = 0$$

In [34]:
```
mom = D(Ku).trace(0,2) - D(trK)
print(mom)
mom.display()
```

```
1-form on the 3-dimensional differentiable manifold Sigma
```

Out[34]: 0

Dynamical Einstein equations

Let us first evaluate the symmetric bilinear form $k_{ij} := K_{ik} K^k{}_j$:

In [35]:
```
KK = K['_ik']*Ku['^k_j']
print(KK)
```

```
Tensor field of type (0,2) on the 3-dimensional differentiable manifold
 Sigma
```

In [36]:
```
KK1 = KK.symmetrize()
KK == KK1
```

Out[36]: True

In [37]:
```
KK = KK1
print(KK)
```

```
Field of symmetric bilinear forms on the 3-dimensional differentiable m
anifold Sigma
```

In [38]:
```
KK[1,1]
```

Out[38]:

$$\frac{a^6 r^4 + 6\,a^4 r^6 + 9\,a^2 r^8 - \left(a^{10} - 2\,a^8 r^2 + a^6 r^4\right) y^6 + \left(a^{10} + 5\,a^6 r^4 - 6\,a^4 r^6\right) y^4 - \left(2\,a^8 r^2 + 5\,a^6 r^4 + 9\,a^2 r^8\right) y^2}{\begin{array}{l} 3\,a^2 r^{12} + r^{14} + 6\,a^4 r^9 - 2\,r^{13} + 4\,a^6 r^6 + \left(3\,a^4 - 8\,a^2\right) r^{10} + \left(a^6 - 4\,a^4\right) r^8 \\ + \left(a^{14} + a^8 r^6 - 6\,a^{12} r - 6\,a^8 r^5 + 3\,\left(a^{10} + 4\,a^8\right) r^4 - 4\,\left(3\,a^{10} + 2\,a^8\right) r^3 + 3 y^8 \right. \\ \quad \left. \left(a^{12} + 4\,a^{10}\right) r^2\right) \\ + 4\,\left(a^6 - 2\,a^4\right) r^7 + 4 \\ \left(a^6 r^8 + a^{12} r - 5\,a^6 r^7 + \left(3\,a^8 + 8\,a^6\right) r^6 - \left(9\,a^8 + 4\,a^6\right) r^5 + \left(3\,a^{10} + 4\,a^8\right) r^4 y^6 \right. \\ \left. - \left(3\,a^{10} - 4\,a^8\right) r^3 + \left(a^{12} - 4\,a^{10}\right) r^2\right) \\ + 2 \\ \left(3\,a^4 r^{10} - 12\,a^4 r^9 + 2\,a^{10} r^2 + 16\,a^6 r^5 + \left(9\,a^6 + 14\,a^4\right) r^8 - 2\,\left(9\,a^6 + 2\,a^4\right) r^7 y^4 \right. \\ \left. + 3\,\left(3\,a^8 - 2\,a^6\right) r^6 + 3\,\left(a^{10} - 6\,a^8\right) r^4 + 2\,\left(3\,a^{10} - 2\,a^8\right) r^3\right) \\ + 4\,\left(a^2 r^{12} - 3\,a^4 r^9 - 3\,a^2 r^{11} + 2\,a^8 r^4 + \left(3\,a^4 + 2\,a^2\right) r^{10} + 3\,\left(a^6 - 2\,a^4\right) r^8 y^2 \right. \\ \quad \left. + \left(3\,a^6 + 4\,a^4\right) r^7 + \left(a^8 - 6\,a^6\right) r^6 + \left(3\,a^8 - 4\,a^6\right) r^5\right) \end{array}}$$

In [39]:
```
KK[1,2]
```

Out[39]:

$$\frac{2\left(\left(a^{8}r-a^{6}r^{3}\right)y^{5}-\left(a^{8}r+3\,a^{4}r^{5}\right)y^{3}+\left(a^{6}r^{3}+3\,a^{4}r^{5}\right)y\right)}{a^{4}r^{8}+2\,a^{2}r^{10}+r^{12}+4\,a^{4}r^{7}+4\,a^{2}r^{9}+4\,a^{4}r^{6}+\left(a^{12}+a^{8}r^{4}-4\,a^{10}r-4\,a^{8}r^{3}+2\left(a^{10}+2\,a^{8}\right)r^{2}\right)y^{8}+4\left(a^{6}r^{6}+a^{10}r-2\,a^{8}r^{3}-3\,a^{6}r^{5}+2\left(a^{8}+a^{6}\right)r^{4}+\left(a^{10}-2\,a^{8}\right)r^{2}\right)y^{6}+2\left(3\,a^{4}r^{8}+6\,a^{8}r^{3}-6\,a^{4}r^{7}+2\,a^{8}r^{2}+2\left(3\,a^{6}+a^{4}\right)r^{6}+\left(3\,a^{8}-8\,a^{6}\right)r^{4}\right)y^{4}+4\left(2\,a^{4}r^{8}+a^{2}r^{10}+3\,a^{6}r^{5}+2\,a^{4}r^{7}-a^{2}r^{9}+2\,a^{6}r^{4}+\left(a^{6}-2\,a^{4}\right)r^{6}\right)y^{2}}$$

In [40]:
```
KK[1,3]
```

Out[40]: 0

In [41]:
```
KK[2,2]
```

Out[41]:

$$-\frac{4\left(\left(a^{8}r^{2}+a^{6}r^{4}-2\,a^{6}r^{3}\right)y^{4}-\left(a^{8}r^{2}+a^{6}r^{4}-2\,a^{6}r^{3}\right)y^{2}\right)}{a^{4}r^{8}+2\,a^{2}r^{10}+r^{12}+4\,a^{4}r^{7}+4\,a^{2}r^{9}+4\,a^{4}r^{6}+\left(a^{12}+a^{8}r^{4}-4\,a^{10}r-4\,a^{8}r^{3}+2\left(a^{10}+2\,a^{8}\right)r^{2}\right)y^{8}+4\left(a^{6}r^{6}+a^{10}r-2\,a^{8}r^{3}-3\,a^{6}r^{5}+2\left(a^{8}+a^{6}\right)r^{4}+\left(a^{10}-2\,a^{8}\right)r^{2}\right)y^{6}+2\left(3\,a^{4}r^{8}+6\,a^{8}r^{3}-6\,a^{4}r^{7}+2\,a^{8}r^{2}+2\left(3\,a^{6}+a^{4}\right)r^{6}+\left(3\,a^{8}-8\,a^{6}\right)r^{4}\right)y^{4}+4\left(2\,a^{4}r^{8}+a^{2}r^{10}+3\,a^{6}r^{5}+2\,a^{4}r^{7}-a^{2}r^{9}+2\,a^{6}r^{4}+\left(a^{6}-2\,a^{4}\right)r^{6}\right)y^{2}}$$

In [42]:
```
KK[2,3]
```

Out[42]: 0

In [43]:
```
KK[3,3]
```

Out[43]:

$$\frac{a^{6}r^{4}+6\,a^{4}r^{6}+9\,a^{2}r^{8}+\left(a^{10}-6\,a^{8}r^{2}-3\,a^{6}r^{4}+8\,a^{6}r^{3}\right)y^{8}-2\left(a^{10}-7\,a^{8}r^{2}-3\,a^{6}r^{4}-3\,a^{4}r^{6}+12\,a^{6}r^{3}\right)y^{6}+\left(a^{10}-10\,a^{8}r^{2}-2\,a^{6}r^{4}-6\,a^{4}r^{6}+9\,a^{2}r^{8}+24\,a^{6}r^{3}\right)y^{4}+2\left(a^{8}r^{2}-a^{6}r^{4}-3\,a^{4}r^{6}-9\,a^{2}r^{8}-4\,a^{6}r^{3}\right)y^{2}}{a^{2}r^{10}+r^{12}+2\,a^{2}r^{9}+\left(a^{12}+a^{10}r^{2}-2\,a^{10}r\right)y^{10}+\left(5\,a^{10}r^{2}+5\,a^{8}r^{4}+2\,a^{10}r-8\,a^{8}r^{3}\right)y^{8}+2\left(5\,a^{8}r^{4}+5\,a^{6}r^{6}+4\,a^{8}r^{3}-6\,a^{6}r^{5}\right)y^{6}+2\left(5\,a^{6}r^{6}+5\,a^{4}r^{8}+6\,a^{6}r^{5}-4\,a^{4}r^{7}\right)y^{4}+\left(5\,a^{4}r^{8}+5\,a^{2}r^{10}+8\,a^{4}r^{7}-2\,a^{2}r^{9}\right)y^{2}}$$

In vacuum and for stationary spacetimes, the dynamical Einstein equations are

$$\mathcal{L}_{\beta}K_{ij}-D_iD_jN+N\left(R_{ij}+KK_{ij}-2K_{ik}K^{k}{}_{j}\right)=0$$

In [44]:
```
dyn = K.lie_der(b) - D(D(N)) + N*( Ric + trK*K - 2*KK )
print(dyn)
dyn.display()
```

```
Tensor field of type (0,2) on the 3-dimensional differentiable manifold
 Sigma
```

Out[44]: 0

Hence, we have checked that all the vacuum 3+1 Einstein equations are fulfilled.

Electric and magnetic parts of the Weyl tensor

The electric part is the bilinear form E given by

$$E_{ij} = R_{ij} + KK_{ij} - K_{ik}K^{k}_{\ j}$$

In [45]:
```
E = Ric + trK*K - KK
print(E)
```

```
Field of symmetric bilinear forms on the 3-dimensional differentiable m
anifold Sigma
```

In [46]:
```
E[1,1]
```

Out[46]:

$$-\frac{3\,a^{4}r^{3} + 5\,a^{2}r^{5} + 2\,r^{7} - 2\,a^{2}r^{4} + 3\left(a^{6}r + a^{4}r^{3} - 2\,a^{4}r^{2}\right)y^{4} - \left(9\,a^{6}r + 16\,a^{4}r^{3} + 7\,a^{2}r^{5} - 6\,a^{4}r^{2} - 2\,a^{2}r^{4}\right)y^{2}}{2\,a^{2}r^{8} + r^{10} + 2\,a^{4}r^{5} - 2\,r^{9} + \left(a^{4} - 4\,a^{2}\right)r^{6} + \left(a^{10} + a^{6}r^{4} - 4\,a^{8}r - 4\,a^{6}r^{3} + 2\left(a^{8} + 2\,a^{6}\right)r^{2}\right)y^{6} + \left(3\,a^{4}r^{6} + 2\,a^{8}r - 8\,a^{6}r^{3} - 10\,a^{4}r^{5} + 2\left(3\,a^{6} + 4\,a^{4}\right)r^{4} + \left(3\,a^{8} - 4\,a^{6}\right)r^{2}\right)y^{4} + \left(3\,a^{2}r^{8} + 4\,a^{6}r^{3} - 4\,a^{4}r^{5} - 8\,a^{2}r^{7} + 2\left(3\,a^{4} + 2\,a^{2}\right)r^{6} + \left(3\,a^{6} - 8\,a^{4}\right)r^{4}\right)y^{2}}$$

In [47]:
```
E[1,1].factor()
```

Out[47]:

$$-\frac{\left(a^{4}y^{2} + a^{2}r^{2}y^{2} - 2\,a^{2}ry^{2} - 3\,a^{4} - 5\,a^{2}r^{2} - 2\,r^{4} + 2\,a^{2}r\right)\left(3\,a^{2}y^{2} - r^{2}\right)r}{\left(a^{4}y^{2} + a^{2}r^{2}y^{2} - 2\,a^{2}ry^{2} + a^{2}r^{2} + r^{4} + 2\,a^{2}r\right)\left(a^{2}y^{2} + r^{2}\right)^{2}\left(a^{2} + r^{2} - 2\,r\right)}$$

In [48]:
```
E[1,2]
```

Out[48]:

$$\frac{3\left(\left(a^{6} + a^{4}r^{2}\right)y^{3} - 3\left(a^{4}r^{2} + a^{2}r^{4}\right)y\right)}{a^{2}r^{6} + r^{8} + 2\,a^{2}r^{5} + \left(a^{8} + a^{6}r^{2} - 2\,a^{6}r\right)y^{6} + \left(3\,a^{6}r^{2} + 3\,a^{4}r^{4} + 2\,a^{6}r - 4\,a^{4}r^{3}\right)y^{4} + \left(3\,a^{4}r^{4} + 3\,a^{2}r^{6} + 4\,a^{4}r^{3} - 2\,a^{2}r^{5}\right)y^{2}}$$

In [49]:
```
E[1,2].factor()
```

Out[49]:

$$\frac{3\left(a^{2}y^{2} - 3\,r^{2}\right)\left(a^{2} + r^{2}\right)a^{2}y}{\left(a^{4}y^{2} + a^{2}r^{2}y^{2} - 2\,a^{2}ry^{2} + a^{2}r^{2} + r^{4} + 2\,a^{2}r\right)\left(a^{2}y^{2} + r^{2}\right)^{2}}$$

In [50]:
```
E[1,3]
```

Out[50]: 0

In [51]:
```
E[2,2]
```

Out[51]:

$$-\frac{3\,a^4r^3+4\,a^2r^5+r^7-4\,a^2r^4+6\left(a^6r+a^4r^3-2\,a^4r^2\right)y^4-\left(9\,a^6r+14\,a^4r^3+5\,a^2r^5-12\,a^4r^2-4\,a^2r^4\right)y^2}{\left(a^8+a^6r^2-2\,a^6r\right)y^8-a^2r^6-r^8-2\,a^2r^5-\left(a^8-2\,a^6r^2-3\,a^4r^4-4\,a^6r+4\,a^4r^3\right)y^6-\left(3\,a^6r^2-3\,a^2r^6+2\,a^6r-8\,a^4r^3+2\,a^2r^5\right)y^4-\left(3\,a^4r^4+2\,a^2r^6-r^8+4\,a^4r^3-4\,a^2r^5\right)y^2}$$

In [52]:
```
E[2,2].factor()
```

Out[52]:

$$-\frac{\left(2\,a^4y^2+2\,a^2r^2y^2-4\,a^2ry^2-3\,a^4-4\,a^2r^2-r^4+4\,a^2r\right)\left(3\,a^2y^2-r^2\right)r}{\left(a^4y^2+a^2r^2y^2-2\,a^2ry^2+a^2r^2+r^4+2\,a^2r\right)\left(a^2y^2+r^2\right)^2(y+1)(y-1)}$$

In [53]:
```
E[2,3]
```

Out[53]: 0

In [54]:
```
E[3,3]
```

Out[54]:

$$\frac{a^2r^5+r^7+3\left(a^6r+a^4r^3-2\,a^4r^2\right)y^6+2\,a^2r^4-\left(3\,a^6r+a^4r^3-2\,a^2r^5-12\,a^4r^2-2\,a^2r^4\right)y^4-\left(2\,a^4r^3+3\,a^2r^5+r^7+6\,a^4r^2+4\,a^2r^4\right)y^2}{a^8y^8+4\,a^6r^2y^6+6\,a^4r^4y^4+4\,a^2r^6y^2+r^8}$$

In [55]:
```
E[3,3].factor()
```

Out[55]:

$$\frac{\left(a^4y^2+a^2r^2y^2-2\,a^2ry^2+a^2r^2+r^4+2\,a^2r\right)\left(3\,a^2y^2-r^2\right)r(y+1)(y-1)}{\left(a^2y^2+r^2\right)^4}$$

The magnetic part is the bilinear form B defined by

$$B_{ij}=\epsilon^k{}_{li}D_kK^l{}_j,$$

where $\epsilon^k{}_{li}$ are the components of the type-(1,2) tensor $\epsilon^\sharp$, related to the Levi-Civita alternating tensor ϵ associated with γ by $\epsilon^k{}_{li}=\gamma^{km}\epsilon_{mli}$. In SageManifolds, ϵ is obtained by the command `volume_form()` and $\epsilon^\sharp$ by the command `volume_form(1)` (1 = 1 index raised):

In [56]:
```
eps = gam.volume_form()
print(eps) ; eps.display()
```

```
3-form eps_gam on the 3-dimensional differentiable manifold Sigma
```

Out[56]:

$$\epsilon_\gamma=\left(\frac{\sqrt{a^2r^2+r^4+2\,a^2r+\left(a^4+a^2r^2-2\,a^2r\right)y^2}\sqrt{a^2y^2+r^2}}{\sqrt{a^2+r^2-2\,r}}\right)\mathrm{d}r\wedge\mathrm{d}y\wedge\mathrm{d}\phi$$

In [57]:
```
epsu = gam.volume_form(1)
print(epsu) ; epsu.display()
```

```
Tensor field of type (1,2) on the 3-dimensional differentiable manifold
 Sigma
```

Out[57]:

$$\left(\frac{\sqrt{a^2r^2 + r^4 + 2\,a^2r + \left(a^4 + a^2r^2 - 2\,a^2r\right)y^2}\sqrt{a^2 + r^2 - 2\,r}}{\sqrt{a^2y^2 + r^2}}\right)\frac{\partial}{\partial r} \otimes \mathrm{d}y \otimes \mathrm{d}$$
$$+\left(-\frac{\sqrt{a^2r^2 + r^4 + 2\,a^2r + \left(a^4 + a^2r^2 - 2\,a^2r\right)y^2}\sqrt{a^2 + r^2 - 2\,r}}{\sqrt{a^2y^2 + r^2}}\right)\frac{\partial}{\partial r} \otimes \mathrm{d}\phi$$
$$\otimes \mathrm{d}y + \left(\frac{\sqrt{a^2r^2 + r^4 + 2\,a^2r + \left(a^4 + a^2r^2 - 2\,a^2r\right)y^2}\left(y^2 - 1\right)}{\sqrt{a^2y^2 + r^2}\sqrt{a^2 + r^2 - 2\,r}}\right)\frac{\partial}{\partial y} \otimes \mathrm{d}r$$
$$\otimes \mathrm{d}\phi + \left(-\frac{\sqrt{a^2r^2 + r^4 + 2\,a^2r + \left(a^4 + a^2r^2 - 2\,a^2r\right)y^2}\left(y^2 - 1\right)}{\sqrt{a^2y^2 + r^2}\sqrt{a^2 + r^2 - 2\,r}}\right)\frac{\partial}{\partial y} \otimes \mathrm{d}\phi$$
$$\otimes \mathrm{d}r$$
$$+\left(-\frac{\sqrt{a^2r^2 + r^4 + 2\,a^2r + \left(a^4 + a^2r^2 - 2\,a^2r\right)y^2}\left(a^2y^2 + r^2\right)^{\frac{3}{2}}}{\left(\left(a^4 + a^2r^2 - 2\,a^2r\right)y^4 - a^2r^2 - r^4 - 2\,a^2r - \left(a^4 - r^4 - 4\,a^2r\right)y^2\right)\sqrt{a^2 + r^2 -}}\right.$$
$$\otimes \mathrm{d}r \otimes \mathrm{d}y$$
$$+\left(\frac{\sqrt{a^2r^2 + r^4 + 2\,a^2r + \left(a^4 + a^2r^2 - 2\,a^2r\right)y^2}\left(a^2y^2 + r^2\right)^{\frac{3}{2}}}{\left(\left(a^4 + a^2r^2 - 2\,a^2r\right)y^4 - a^2r^2 - r^4 - 2\,a^2r - \left(a^4 - r^4 - 4\,a^2r\right)y^2\right)\sqrt{a^2 + r^2 -}}\right.$$
$$\otimes \mathrm{d}y \otimes \mathrm{d}r$$

In [58]:
```
DKu = D(Ku)
B = epsu['^k_li']*DKu['^l_jk']
print(B)
```

```
Tensor field of type (0,2) on the 3-dimensional differentiable manifold
 Sigma
```

Let us check that B is symmetric:

In [59]:
```
B1 = B.symmetrize()
B == B1
```

Out[59]: True

Accordingly, we set

In [60]:
```
B = B1
B.set_name('B')
print(B)
```

```
Field of symmetric bilinear forms B on the 3-dimensional differentiable
 manifold Sigma
```

In [61]:
```
B[1,1]
```

Out[61]:

$$-\frac{\left(a^{7} + a^{5} r^{2} - 2 \, a^{5} r\right) y^{5} - \left(3 \, a^{7} + 8 \, a^{5} r^{2} + 5 \, a^{3} r^{4} - 2 \, a^{5} r - 6 \, a^{3} r^{3}\right) y^{3} + 3 \left(3 \, a^{5} r^{2} + 5 \, a^{3} r^{4} + 2 \, a r^{6} - 2 \, a^{3} r^{3}\right) y}{2 \, a^{2} r^{8} + r^{10} + 2 \, a^{4} r^{5} - 2 \, r^{9} + \left(a^{4} - 4 \, a^{2}\right) r^{6} + \left(a^{10} + a^{6} r^{4} - 4 \, a^{8} r - 4 \, a^{6} r^{3} + 2 \, \left(a^{8} + 2 \, a^{6}\right) r^{2}\right) y^{6} + \left(3 \, a^{4} r^{6} + 2 \, a^{8} r - 8 \, a^{6} r^{3} - 10 \, a^{4} r^{5} + 2 \, \left(3 \, a^{6} + 4 \, a^{4}\right) r^{4} + \left(3 \, a^{8} - 4 \, a^{6}\right) r^{2}\right) y^{4} + \left(3 \, a^{2} r^{8} + 4 \, a^{6} r^{3} - 4 \, a^{4} r^{5} - 8 \, a^{2} r^{7} + 2 \, \left(3 \, a^{4} + 2 \, a^{2}\right) r^{6} + \left(3 \, a^{6} - 8 \, a^{4}\right) r^{4}\right) y^{2}}$$

In [62]:
```
B[1,1].factor()
```

Out[62]:

$$-\frac{\left(a^{4} y^{2} + a^{2} r^{2} y^{2} - 2 \, a^{2} r y^{2} - 3 \, a^{4} - 5 \, a^{2} r^{2} - 2 \, r^{4} + 2 \, a^{2} r\right) \left(a^{2} y^{2} - 3 \, r^{2}\right) a y}{\left(a^{4} y^{2} + a^{2} r^{2} y^{2} - 2 \, a^{2} r y^{2} + a^{2} r^{2} + r^{4} + 2 \, a^{2} r\right) \left(a^{2} y^{2} + r^{2}\right)^{2} \left(a^{2} + r^{2} - 2 \, r\right)}$$

In [63]:
```
B[1,2]
```

Out[63]:

$$\frac{3 \, \left(a^{3} r^{3} + a r^{5} - 3 \, \left(a^{5} r + a^{3} r^{3}\right) y^{2}\right)}{a^{2} r^{6} + r^{8} + 2 \, a^{2} r^{5} + \left(a^{8} + a^{6} r^{2} - 2 \, a^{6} r\right) y^{6} + \left(3 \, a^{6} r^{2} + 3 \, a^{4} r^{4} + 2 \, a^{6} r - 4 \, a^{4} r^{3}\right) y^{4} + \left(3 \, a^{4} r^{4} + 3 \, a^{2} r^{6} + 4 \, a^{4} r^{3} - 2 \, a^{2} r^{5}\right) y^{2}}$$

In [64]:
```
B[1,2].factor()
```

Out[64]:

$$-\frac{3 \, \left(3 \, a^{2} y^{2} - r^{2}\right) \left(a^{2} + r^{2}\right) a r}{\left(a^{4} y^{2} + a^{2} r^{2} y^{2} - 2 \, a^{2} r y^{2} + a^{2} r^{2} + r^{4} + 2 \, a^{2} r\right) \left(a^{2} y^{2} + r^{2}\right)^{2}}$$

In [65]:
```
B[1,3]
```

Out[65]: 0

In [66]:
```
B[2,2]
```

Out[66]:

$$-\frac{2 \, \left(a^{7} + a^{5} r^{2} - 2 \, a^{5} r\right) y^{5} - \left(3 \, a^{7} + 10 \, a^{5} r^{2} + 7 \, a^{3} r^{4} - 4 \, a^{5} r - 12 \, a^{3} r^{3}\right) y^{3} + 3 \left(3 \, a^{5} r^{2} + 4 \, a^{3} r^{4} + a r^{6} - 4 \, a^{3} r^{3}\right) y}{\left(a^{8} + a^{6} r^{2} - 2 \, a^{6} r\right) y^{8} - a^{2} r^{6} - r^{8} - 2 \, a^{2} r^{5} - \left(a^{8} - 2 \, a^{6} r^{2} - 3 \, a^{4} r^{4} - 4 \, a^{6} r + 4 \, a^{4} r^{3}\right) y^{6} - \left(3 \, a^{6} r^{2} - 3 \, a^{2} r^{6} + 2 \, a^{6} r - 8 \, a^{4} r^{3} + 2 \, a^{2} r^{5}\right) y^{4} - \left(3 \, a^{4} r^{4} + 2 \, a^{2} r^{6} - r^{8} + 4 \, a^{4} r^{3} - 4 \, a^{2} r^{5}\right) y^{2}}$$

In [67]:
```
B[2,2].factor()
```

Out[67]:

$$-\frac{\left(2 \, a^{4} y^{2} + 2 \, a^{2} r^{2} y^{2} - 4 \, a^{2} r y^{2} - 3 \, a^{4} - 4 \, a^{2} r^{2} - r^{4} + 4 \, a^{2} r\right) \left(a^{2} y^{2} - 3 \, r^{2}\right) a y}{\left(a^{4} y^{2} + a^{2} r^{2} y^{2} - 2 \, a^{2} r y^{2} + a^{2} r^{2} + r^{4} + 2 \, a^{2} r\right) \left(a^{2} y^{2} + r^{2}\right)^{2} (y + 1)(y - 1)}$$

In [68]:
```
B[2,3]
```

Out[68]: 0

In [69]:
```
B[3,3]
```

Out[69]:
$$\frac{\left(a^7 + a^5r^2 - 2\,a^5r\right)y^7 - \left(a^7 + 3\,a^5r^2 + 2\,a^3r^4 - 4\,a^5r - 6\,a^3r^3\right)y^5 + \left(2\,a^5r^2 - a^3r^4 - 3\,ar^6 - 2\,a^5r - 12\,a^3r^3\right)y^3 + 3\left(a^3r^4 + ar^6 + 2\,a^3r^3\right)y}{a^8y^8 + 4\,a^6r^2y^6 + 6\,a^4r^4y^4 + 4\,a^2r^6y^2 + r^8}$$

In [70]:
```
B[3,3].factor()
```

Out[70]:
$$\frac{\left(a^4y^2 + a^2r^2y^2 - 2\,a^2ry^2 + a^2r^2 + r^4 + 2\,a^2r\right)\left(a^2y^2 - 3\,r^2\right)a(y+1)(y-1)y}{\left(a^2y^2 + r^2\right)^4}$$

3+1 decomposition of the Simon-Mars tensor

We follow the computation presented in arXiv:1412.6542. We start by the tensor $E^\sharp$ of components $E^i{}_j$:

In [71]:
```
Eu = E.up(gam, 0)
print(Eu)
```

```
Tensor field of type (1,1) on the 3-dimensional differentiable manifold
 Sigma
```

Tensor $B^\sharp$ of components $B^i{}_j$:

In [72]:
```
Bu = B.up(gam, 0)
print(Bu)
```

```
Tensor field of type (1,1) on the 3-dimensional differentiable manifold
 Sigma
```

1-form $\beta^\flat$ of components β_i and its exterior derivative:

In [73]:
```
bd = b.down(gam)
xdb = bd.exterior_derivative()
print(xdb) ; xdb.display()
```

```
2-form on the 3-dimensional differentiable manifold Sigma
```

Out[73]:
$$\left(\frac{2\left(a^3y^4 + ar^2 - \left(a^3 + ar^2\right)y^2\right)}{a^4y^4 + 2\,a^2r^2y^2 + r^4}\right)\mathrm{d}r\wedge\mathrm{d}\phi + \left(\frac{4\left(a^3r + ar^3\right)y}{a^4y^4 + 2\,a^2r^2y^2 + r^4}\right)\mathrm{d}y\wedge\mathrm{d}\phi$$

Scalar square of shift $\beta_i\beta^i$:

In [74]:
```
b2 = bd(b)
print(b2) ; b2.display()
```

```
Scalar field on the 3-dimensional differentiable manifold Sigma
```

Out[74]:
$$\begin{array}{llcl} & \Sigma & \longrightarrow & \mathbb{R} \\ & (r,y,\phi) & \longmapsto & -\frac{4\left(a^2r^2y^2 - a^2r^2\right)}{a^2r^4 + r^6 + 2\,a^2r^3 + \left(a^6 + a^4r^2 - 2\,a^4r\right)y^4 + 2\left(a^4r^2 + a^2r^4 + a^4r - a^2r^3\right)y^2} \end{array}$$

Scalar $Y = E(\beta,\beta) = E_{ij}\beta^i\beta^j$:

In [75]:
```
Ebb = E(b,b)
Y = Ebb
print(Y) ; Y.display()
```

```
Scalar field on the 3-dimensional differentiable manifold Sigma
```

Out[75]:

$$\begin{array}{llll} \Sigma & \longrightarrow & \mathbb{R} \\ (r, y, \phi) & \longmapsto & \frac{4\,\left(3\,a^4 r^3 y^4 + a^2 r^5 - \left(3\,a^4 r^3 + a^2 r^5\right) y^2\right)}{a^2 r^{10} + r^{12} + 2\,a^2 r^9 + \left(a^{12} + a^{10} r^2 - 2\,a^{10} r\right) y^{10} + \left(5\,a^{10} r^2 + 5\,a^8 r^4 + 2\,a^{10} r - 8\,a^8 r^3\right) y^8 + 2\,\left(5\,a^8 r^4 + 5\,a^6 r^6 + 4\,a^8 r^3 - 6\,a^6 r^5\right) y^6 + 2\,\left(5\,a^6 r^6 + 5\,a^4 r^8 + 6\,a^6 r^5 - 4\,a^4 r^7\right) y^4 + \left(5\,a^4 r^8 + 5\,a^2 r} \end{array}$$

In [76]:
```
Ebb.coord_function().factor()
```

Out[76]:

$$\frac{4\,\left(3\,a^2 y^2 - r^2\right) a^2 r^3 (y+1)(y-1)}{\left(a^4 y^2 + a^2 r^2 y^2 - 2\,a^2 r y^2 + a^2 r^2 + r^4 + 2\,a^2 r\right)\left(a^2 y^2 + r^2\right)^4}$$

In [77]:
```
Ebb.display()
```

Out[77]:

$$\begin{array}{llll} \Sigma & \longrightarrow & \mathbb{R} \\ (r, y, \phi) & \longmapsto & \frac{4\,\left(3\,a^2 y^2 - r^2\right) a^2 r^3 (y+1)(y-1)}{\left(a^4 y^2 + a^2 r^2 y^2 - 2\,a^2 r y^2 + a^2 r^2 + r^4 + 2\,a^2 r\right)\left(a^2 y^2 + r^2\right)^4} \end{array}$$

Scalar $\bar{Y} = B(\beta, \beta) = B_{ij}\beta^i\beta^j$:

In [78]:
```
Bbb = B(b,b)
Y_bar = Bbb
print(Y_bar) ; Y_bar.display()
```

```
Scalar field B(beta,beta) on the 3-dimensional differentiable manifold
Sigma
```

Out[78]:

$$\begin{array}{llll} B(\beta,\beta): & \Sigma & \longrightarrow & \mathbb{R} \\ & (r, y, \phi) & \longmapsto & \frac{4\,\left(a^5 r^2 y^5 + 3\,a^3 r^4 y - \left(a^5 r^2 + 3\,a^3 r^4\right) y^3\right)}{a^2 r^{10} + r^{12} + 2\,a^2 r^9 + \left(a^{12} + a^{10} r^2 - 2\,a^{10} r\right) y^{10} + \left(5\,a^{10} r^2 + 5\,a^8 r^4 + 2\,a^{10} r - 8\,a^8 r^3\right) y^8 + 2\,\left(5\,a^8 r^4 + 5\,a^6 r^6 + 4\,a^8 r^3 - 6\,a^6 r^5\right) y^6 + 2\,\left(5\,a^6 r^6 + 5\,a^4 r^8 + 6\,a^6 r^5 - 4\,a^4 r^7\right) y^4 + \left(5\right.} \end{array}$$

In [79]:
```
Bbb.coord_function().factor()
```

Out[79]:

$$\frac{4\,\left(a^2 y^2 - 3\,r^2\right) a^3 r^2 (y+1)(y-1) y}{\left(a^4 y^2 + a^2 r^2 y^2 - 2\,a^2 r y^2 + a^2 r^2 + r^4 + 2\,a^2 r\right)\left(a^2 y^2 + r^2\right)^4}$$

1-form of components $Eb_i = E_{ij}\beta^j$:

In [80]:
```
Eb = E.contract(b)
print(Eb) ; Eb.display()
```

```
1-form on the 3-dimensional differentiable manifold Sigma
```

Out[80]:

$$\left(-\frac{2\,\left(3\,a^3 r^2 y^4 + a r^4 - \left(3\,a^3 r^2 + a r^4\right) y^2\right)}{a^8 y^8 + 4\,a^6 r^2 y^6 + 6\,a^4 r^4 y^4 + 4\,a^2 r^6 y^2 + r^8}\right) \mathrm{d}\phi$$

Vector field of components $Eub^i = E^i{}_j\beta^j$:

In [81]:
```
Eub = Eu.contract(b)
print(Eub) ; Eub.display()
```

```
Vector field on the 3-dimensional differentiable manifold Sigma
```

Out[81]:

$$\left(\frac{2\,\left(3\,a^{3}r^{2}y^{2}-ar^{4}\right)}{a^{2}r^{8}+r^{10}+2\,a^{2}r^{7}+\left(a^{10}+a^{8}r^{2}-2\,a^{8}r\right)y^{8}+2\left(2\,a^{8}r^{2}+2\,a^{6}r^{4}+a^{8}r-3\,a^{6}r^{3}\right)y^{6}+6\,\left(a^{6}r^{4}+a^{4}r^{6}+a^{6}r^{3}-a^{4}r^{5}\right)y^{4}+2\left(2\,a^{4}r^{6}+2\,a^{2}r^{8}+3\,a^{4}r^{5}-a^{2}r^{7}\right)y^{2}}\right)\frac{\partial}{\partial\phi}$$

1-form of components $Bb_i = B_{ij}\beta^j$:

In [82]:
```
Bb = B.contract(b)
print(Bb) ; Bb.display()
```

```
1-form on the 3-dimensional differentiable manifold Sigma
```

Out[82]:

$$\left(-\frac{2\,\left(a^{4}ry^{5}+3\,a^{2}r^{3}y-\left(a^{4}r+3\,a^{2}r^{3}\right)y^{3}\right)}{a^{8}y^{8}+4\,a^{6}r^{2}y^{6}+6\,a^{4}r^{4}y^{4}+4\,a^{2}r^{6}y^{2}+r^{8}}\right)\mathrm{d}\phi$$

Vector field of components $Bub^i = B^i{}_j\beta^j$:

In [83]:
```
Bub = Bu.contract(b)
print(Bub) ; Bub.display()
```

```
Vector field on the 3-dimensional differentiable manifold Sigma
```

Out[83]:

$$\left(\frac{2\left(a^4 r y^3 - 3\,a^2 r^3 y\right)}{a^2 r^8 + r^{10} + 2\,a^2 r^7 + \left(a^{10} + a^8 r^2 - 2\,a^8 r\right) y^8 + 2\left(2\,a^8 r^2 + 2\,a^6 r^4 + a^8 r - 3\,a^6 r^3\right) y^6 + 6\left(a^6 r^4 + a^4 r^6 + a^6 r^3 - a^4 r^5\right) y^4 + 2\left(2\,a^4 r^6 + 2\,a^2 r^8 + 3\,a^4 r^5 - a^2 r^7\right) y^2} \right) \frac{\partial}{\partial \phi}$$

Vector field of components $Kub^i = K^i{}_j \beta^j$:

In [84]:
```
Kub = Ku.contract(b)
print(Kub) ; Kub.display()
```

```
Vector field on the 3-dimensional differentiable manifold Sigma
```

Out[84]:

$$-\frac{2\left(a^6 r^3 + 4\,a^4 r^5 + 3\,a^2 r^7 - 2\,a^4 r^4 - 6\,a^2 r^6 + \left(a^8 r - a^4 r^5 - \right.\right. - \left(a^8 r + a^6 r^3 + 3\,a^4 r^5 + 3\,a^2 r^7 - 2\,a^6 r^2 - 6\,a^2 r^6\right) y^2\Big)}{\left(a^2 r^8 + r^{10} + 2\,a^2 r^7 + \left(a^{10} + a^8 r^2 - 2\,a^8 r\right) y^8 + 2\left(2\,a^8 r^2 + 2\,a^6 r^4 + a^8 r - 3\,a^6 r^3\right) y^6 + 6\left(a^6 r^4 + a^4 r^6 + a^6 r^3 - a^4 r^5\right) y^4 + 2\left(2\,a^4 r^6 + 2\,a^2 r^8 + 3\,a^4 r^5 - a^2 r^7\right) y^2\right) \sqrt{a^2 r^2}}$$

In [85]:
```
T = 2*b(N) - 2*K(b,b)
print(T) ; T.display()
```

```
Scalar field zero on the 3-dimensional differentiable manifold Sigma
```

Out[85]:

$$\begin{array}{llcl} 0: & \Sigma & \longrightarrow & \mathbb{R} \\ & (r, y, \phi) & \longmapsto & 0 \end{array}$$

In [86]:
```
Db = D(b)  #  Db^i_j = D_j b^i
Dbu = Db.up(gam, 1)  # Dbu^{ij} = D^j b^i
bDb = b*Dbu  # bDb^{ijk} = b^i D^k b^j
T_bar = eps['_ijk']*bDb['^ikj']
print(T_bar) ; T_bar.display()
```

```
Scalar field zero on the 3-dimensional differentiable manifold Sigma
```

Out[86]:

$$\begin{array}{llcl} 0: & \Sigma & \longrightarrow & \mathbb{R} \\ & (r, y, \phi) & \longmapsto & 0 \end{array}$$

In [87]:
```
epsb = eps.contract(b)
print(epsb)
epsb.display()
```

```
2-form on the 3-dimensional differentiable manifold Sigma
```

Out[87]:

$$\left(-\frac{2\sqrt{a^2y^2+r^2}ar}{\sqrt{a^2r^2+r^4+2a^2r+\left(a^4+a^2r^2-2a^2r\right)y^2}\sqrt{a^2+r^2-2r}}\right)\mathrm{d}r\wedge\mathrm{d}y$$

In [88]:
```
epsB = eps['_ijl']*Bu['^l_k']
print(epsB)
```

```
Tensor field of type (0,3) on the 3-dimensional differentiable manifold
 Sigma
```

In [89]:
```
epsB.symmetries()
```

```
no symmetry; antisymmetry: (0, 1)
```

In [90]:
```
epsB[1,2,3]
```

Out[90]:

$$-\frac{\left(a^3y^3-3ar^2y\right)\sqrt{a^2r^2+r^4+2a^2r+\left(a^4+a^2r^2-2a^2r\right)y^2}\sqrt{a^2y^2+r^2}}{\left(a^6y^6+3a^4r^2y^4+3a^2r^4y^2+r^6\right)\sqrt{a^2+r^2-2r}}$$

In [91]:
```
Z = 2*N*( D(N)  -K.contract(b)) + b.contract(xdb)
print(Z) ; Z.display()
```

```
1-form on the 3-dimensional differentiable manifold Sigma
```

Out[91]:

$$\left(-\frac{2\left(a^2y^2-r^2\right)}{a^4y^4+2a^2r^2y^2+r^4}\right)\mathrm{d}r+\left(\frac{4a^2ry}{a^4y^4+2a^2r^2y^2+r^4}\right)\mathrm{d}y$$

In [92]:
```
DNu = D(N).up(gam)
A = 2*(DNu - Ku.contract(b))*b + N*Dbu
Z_bar = eps['_ijk']*A['^kj']
print(Z_bar) ; Z_bar.display()
```

```
1-form on the 3-dimensional differentiable manifold Sigma
```

Out[92]:

$$\left(\frac{4ary}{a^4y^4+2a^2r^2y^2+r^4}\right)\mathrm{d}r+\left(\frac{2\left(a^3y^2-ar^2\right)}{a^4y^4+2a^2r^2y^2+r^4}\right)\mathrm{d}y$$

In [93]:
```
# Test:
Dbdu = D(bd).up(gam,1).up(gam,1)  # (Db)^{ij} = D^i b^j
A = 2*b*(DNu - Ku.contract(b)) + N*Dbdu
Z_bar0 = eps['_ijk']*A['^jk']  # NB: '^jk' and not 'kj'
Z_bar0 == Z_bar
```

Out[93]: True

In [94]:
```
W = N*Eb + epsb.contract(Bub)
print(W) ; W.display()
```

1-form on the 3-dimensional differentiable manifold Sigma

Out[94]:

$$\left(-\frac{2\left(3a^3r^2y^4 + ar^4 - \left(3a^3r^2 + ar^4\right)y^2\right)\sqrt{a^2y^2 + r^2}\sqrt{a^2 + r^2 - 2r}}{\left(a^8y^8 + 4a^6r^2y^6 + 6a^4r^4y^4 + 4a^2r^6y^2 + r^8\right)\sqrt{a^2r^2 + r^4 + 2a^2r + \left(a^4 + a^2r^2 - \right.}} \right) \mathrm{d}\phi$$

In [95]:
```
W_bar = N*Bb - epsb.contract(Eub)
print(W_bar) ; W_bar.display()
```

1-form on the 3-dimensional differentiable manifold Sigma

Out[95]:

$$\left(-\frac{2\left(a^4ry^5 + 3a^2r^3y - \left(a^4r + 3a^2r^3\right)y^3\right)\sqrt{a^2y^2 + r^2}\sqrt{a^2 + r^2 - 2r}}{\left(a^8y^8 + 4a^6r^2y^6 + 6a^4r^4y^4 + 4a^2r^6y^2 + r^8\right)\sqrt{a^2r^2 + r^4 + 2a^2r + \left(a^4 + a^2r^2 - \right.}} \right) \mathrm{d}\phi$$

In [96]:
```
W[3].factor()
```

Out[96]:

$$-\frac{2\left(3a^2y^2 - r^2\right)\sqrt{a^2 + r^2 - 2r}\,ar^2(y+1)(y-1)}{\sqrt{a^2r^2 + r^4 + 2a^2r + \left(a^4 + a^2r^2 - 2a^2r\right)y^2}\left(a^2y^2 + r^2\right)^{\frac{7}{2}}}$$

In [97]:
```
W_bar[3].factor()
```

Out[97]:

$$-\frac{2\left(a^2y^2 - 3r^2\right)\sqrt{a^2 + r^2 - 2r}\,ra^2r(y+1)(y-1)y}{\sqrt{a^2r^2 + r^4 + 2a^2r + \left(a^4 + a^2r^2 - 2a^2r\right)y^2}\left(a^2y^2 + r^2\right)^{\frac{7}{2}}}$$

In [98]:
```
M = - 4*Eb(Kub - DNu) - 2*(epsB['_ij.']*Dbu['^ji'])(b)
print(M) ; M.display()
```

```
Scalar field zero on the 3-dimensional differentiable manifold Sigma
```

Out[98]:
$$\begin{array}{llll} 0: & \Sigma & \longrightarrow & \mathbb{R} \\ & (r,y,\phi) & \longmapsto & 0 \end{array}$$

In [99]:
```
M_bar = 2*(eps.contract(Eub))['_ij']*Dbu['^ji'] - 4*Bb(Kub - DNu)
print(M_bar) ; M_bar.display()
```

```
Scalar field zero on the 3-dimensional differentiable manifold Sigma
```

Out[99]:
$$\begin{array}{llll} 0: & \Sigma & \longrightarrow & \mathbb{R} \\ & (r,y,\phi) & \longmapsto & 0 \end{array}$$

In [100]:
```
A = epsB['_ilk']*b['^l'] + epsB['_ikl']*b['^l'] \
    + Bu['^m_i']*epsb['_mk'] - 2*N*E
xdbE = xdb['_kl']*Eu['^k_i']
L = 2*N*epsB['_kli']*Dbu['^kl'] + 2*xdb['_ij']*Eub['^j'] \
    + 2*xdbE['_li']*b['^l'] + 2*A['_ik']*(Kub - DNu)['^k']
print(L)
```

```
1-form on the 3-dimensional differentiable manifold Sigma
```

In [101]:
```
L[1]
```

Out[101]:
$$-\frac{8\left(5\,a^{4}ry^{4} - 10\,a^{2}r^{3}y^{2} + r^{5}\right)}{a^{10}y^{10} + 5\,a^{8}r^{2}y^{8} + 10\,a^{6}r^{4}y^{6} + 10\,a^{4}r^{6}y^{4} + 5\,a^{2}r^{8}y^{2} + r^{10}}$$

In [102]:
```
L[1].factor()
```

Out[102]:
$$-\frac{8\left(5\,a^{4}y^{4} - 10\,a^{2}r^{2}y^{2} + r^{4}\right)r}{\left(a^{2}y^{2} + r^{2}\right)^{5}}$$

In [103]:
```
L[2]
```

Out[103]:
$$-\frac{8\left(a^{6}y^{5} - 10\,a^{4}r^{2}y^{3} + 5\,a^{2}r^{4}y\right)}{a^{10}y^{10} + 5\,a^{8}r^{2}y^{8} + 10\,a^{6}r^{4}y^{6} + 10\,a^{4}r^{6}y^{4} + 5\,a^{2}r^{8}y^{2} + r^{10}}$$

In [104]:
```
L[2].factor()
```

Out[104]:
$$-\frac{8\left(a^{4}y^{4} - 10\,a^{2}r^{2}y^{2} + 5\,r^{4}\right)a^{2}y}{\left(a^{2}y^{2} + r^{2}\right)^{5}}$$

In [105]:
```
L[3]
```

Out[105]: 0

In [106]:
```
N2pbb = N^2 + b2
V = N2pbb*E - 2*(b.contract(E)*bd).symmetrize() + Ebb*gam \
    + 2*N*(b.contract(epsB).symmetrize())
print(V)
```

```
Field of symmetric bilinear forms on the 3-dimensional differentiable m
anifold Sigma
```

In [107]:
```
V[1,1]
```

Out[107]:

$$-\frac{3 a^4 r y^4 + 3 a^2 r^3 + 2 r^5 - 4 r^4 - \left(9 a^4 r + 7 a^2 r^3 - 12 a^2 r^2\right) y^2}{a^2 r^6 + r^8 - 2 r^7 + \left(a^8 + a^6 r^2 - 2 a^6 r\right) y^6 + 3 \left(a^6 r^2 + a^4 r^4 - 2 a^4 r^3\right) y^4 + 3 \left(a^4 r^4 + a^2 r^6 - 2 a^2 r^5\right) y^2}$$

In [108]:
```
V[1,1].factor()
```

Out[108]:

$$-\frac{\left(3 a^2 y^2 - r^2\right)\left(a^2 y^2 - 3 a^2 - 2 r^2 + 4 r\right) r}{\left(a^2 y^2 + r^2\right)^3 \left(a^2 + r^2 - 2 r\right)}$$

In [109]:
```
V[1,2]
```

Out[109]:

$$\frac{3 \left(a^4 y^3 - 3 a^2 r^2 y\right)}{a^6 y^6 + 3 a^4 r^2 y^4 + 3 a^2 r^4 y^2 + r^6}$$

In [110]:
```
V[1,2].factor()
```

Out[110]:

$$\frac{3 \left(a^2 y^2 - 3 r^2\right) a^2 y}{\left(a^2 y^2 + r^2\right)^3}$$

In [111]:
```
V[1,3]
```

Out[111]: 0

In [112]:
```
V[2,2]
```

Out[112]:

$$-\frac{6 a^4 r y^4 + 3 a^2 r^3 + r^5 - 2 r^4 - \left(9 a^4 r + 5 a^2 r^3 - 6 a^2 r^2\right) y^2}{a^6 y^8 - \left(a^6 - 3 a^4 r^2\right) y^6 - r^6 - 3 \left(a^4 r^2 - a^2 r^4\right) y^4 - \left(3 a^2 r^4 - r^6\right) y^2}$$

In [113]:
```
V[2,2].factor()
```

Out[113]:

$$-\frac{\left(3 a^2 y^2 - r^2\right)\left(2 a^2 y^2 - 3 a^2 - r^2 + 2 r\right) r}{\left(a^2 y^2 + r^2\right)^3 (y + 1)(y - 1)}$$

In [114]:
```
V[2,3]
```

Out[114]: 0

In [115]:
```
V[3,3]
```

Out[115]:

$$\frac{a^2 r^3 + r^5 + 3 \left(a^4 r + a^2 r^3 - 2 a^2 r^2\right) y^4 - 2 r^4 - \left(3 a^4 r + 4 a^2 r^3 + r^5 - 6 a^2 r^2 - 2 r^4\right) y^2}{a^6 y^6 + 3 a^4 r^2 y^4 + 3 a^2 r^4 y^2 + r^6}$$

In [116]:
```
V[3,3].factor()
```

Out[116]:

$$\frac{\left(3 a^2 y^2 - r^2\right)\left(a^2 + r^2 - 2 r\right) r (y + 1)(y - 1)}{\left(a^2 y^2 + r^2\right)^3}$$

In [117]:
```
beps = b.contract(eps)
V_bar = N2pbb*B - 2*(b.contract(B)*bd).symmetrize() + Bbb*gam \
        - 2*N*(beps['_il']*Eu['^l_j']).symmetrize()
print(V_bar)
```

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In [118]:
```
V_bar[1,1]
```

Out[118]:

$$-\frac{a^5y^5 - \left(3\,a^5 + 5\,a^3r^2 - 4\,a^3r\right)y^3 + 3\,\left(3\,a^3r^2 + 2\,ar^4 - 4\,ar^3\right)y}{a^2r^6 + r^8 - 2\,r^7 + \left(a^8 + a^6r^2 - 2\,a^6r\right)y^6 + 3\,\left(a^6r^2 + a^4r^4 - 2\,a^4r^3\right)y^4 + 3\,\left(a^4r^4 + a^2r^6 - 2\,a^2r^5\right)y^2}$$

In [119]:
```
V_bar[1,1].factor()
```

Out[119]:

$$-\frac{\left(a^2y^2 - 3\,a^2 - 2\,r^2 + 4\,r\right)\left(a^2y^2 - 3\,r^2\right)ay}{\left(a^2y^2 + r^2\right)^3\left(a^2 + r^2 - 2\,r\right)}$$

In [120]:
```
V_bar[1,2]
```

Out[120]:

$$-\frac{3\,\left(3\,a^3ry^2 - ar^3\right)}{a^6y^6 + 3\,a^4r^2y^4 + 3\,a^2r^4y^2 + r^6}$$

In [121]:
```
V_bar[1,2].factor()
```

Out[121]:

$$-\frac{3\,\left(3\,a^2y^2 - r^2\right)ar}{\left(a^2y^2 + r^2\right)^3}$$

In [122]:
```
V_bar[1,3]
```

Out[122]: 0

In [123]:
```
V_bar[2,2]
```

Out[123]:

$$-\frac{2\,a^5y^5 - \left(3\,a^5 + 7\,a^3r^2 - 2\,a^3r\right)y^3 + 3\,\left(3\,a^3r^2 + ar^4 - 2\,ar^3\right)y}{a^6y^8 - \left(a^6 - 3\,a^4r^2\right)y^6 - r^6 - 3\,\left(a^4r^2 - a^2r^4\right)y^4 - \left(3\,a^2r^4 - r^6\right)y^2}$$

In [124]:
```
V_bar[2,2].factor()
```

Out[124]:

$$-\frac{\left(2\,a^2y^2 - 3\,a^2 - r^2 + 2\,r\right)\left(a^2y^2 - 3\,r^2\right)ay}{\left(a^2y^2 + r^2\right)^3(y+1)(y-1)}$$

In [125]:
```
V_bar[2,3]
```

Out[125]: 0

In [126]:
```
V_bar[3,3]
```

Out[126]:

$$\frac{\left(a^5 + a^3r^2 - 2\,a^3r\right)y^5 - \left(a^5 + 4\,a^3r^2 + 3\,ar^4 - 2\,a^3r - 6\,ar^3\right)y^3 + 3\,\left(a^3r^2 + ar^4 - 2\,ar^3\right)y}{a^6y^6 + 3\,a^4r^2y^4 + 3\,a^2r^4y^2 + r^6}$$

In [127]:
```
V_bar[3,3].factor()
```

Out[127]:

$$\frac{(a^2y^2 - 3r^2)(a^2 + r^2 - 2r)a(y+1)(y-1)y}{(a^2y^2 + r^2)^3}$$

In [128]:
```
G = (N^2 - b2)*gam + bd*bd
print(G)
```

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In [129]:
```
G.display()
```

Out[129]:

$$\left(\frac{a^2y^2 + r^2 - 2r}{a^2 + r^2 - 2r}\right) \mathrm{d}r \otimes \mathrm{d}r + \left(-\frac{a^2y^2 + r^2 - 2r}{y^2 - 1}\right) \mathrm{d}y \otimes \mathrm{d}y + \left(-(a^2 + r^2 - 2r)y^2 + a^2 + r^2 - 2r\right) \mathrm{d}\phi \otimes \mathrm{d}\phi$$

3+1 decomposition of the real part of the Simon-Mars tensor

We follow Eqs. (77)-(80) of arXiv:1412.6542:

In [130]:
```
S1 = (4*(V*Z - V_bar*Z_bar) + G*L).antisymmetrize(1,2)
print(S1)
```

```
Tensor field of type (0,3) on the 3-dimensional differentiable manifold
 Sigma
```

In [131]:
```
S1.display()
```

Out[131]: 0

In [132]:
```
S2 = 4*(T*V - T_bar*V_bar - W*Z + W_bar*Z_bar) + M*G \
      - N*bd*L
print(S2)
```

```
Tensor field of type (0,2) on the 3-dimensional differentiable manifold
 Sigma
```

In [133]:
```
S2.display()
```

Out[133]: 0

In [134]:
```
S3 = (4*(W*Z - W_bar*Z_bar) + N*bd*L).antisymmetrize()
print(S3)
```

```
2-form on the 3-dimensional differentiable manifold Sigma
```

In [135]:
```
S3.display()
```

Out[135]: 0

In [136]:
```
S2[3,1] == -2*S3[3,1]
```

Out[136]: True

In [137]:
```
S2[3,2] == -2*S3[3,2]
```

Out[137]: True

In [138]:
```
S4 = 4*(T*W - T_bar*W_bar) -4*(Y*Z - Y_bar*Z_bar) + N*M*bd \
     - b2*L
print(S4)
```

```
1-form on the 3-dimensional differentiable manifold Sigma
```

In [139]:
```
S4.display()
```

Out[139]: 0

Hence all the tensors S^1, S^2, S^3 and S^4 involved in the 3+1 decomposition of the real part of the Simon-Mars are zero, as they should since the Simon-Mars tensor vanishes identically for the Kerr spacetime.

3+1 decomposition of the imaginary part of the Simon-Mars tensor

We follow Eqs. (82)-(85) of arXiv:1412.6542.

In [140]:
```
epsE = eps['_ijl']*Eu['^l_k']
print(epsE)
```

```
Tensor field of type (0,3) on the 3-dimensional differentiable manifold
 Sigma
```

In [141]:
```
A = - epsE['_ilk']*b['^l'] - epsE['_ikl']*b['^l'] \
     - Eu['^m_i']*epsb['_mk'] - 2*N*B
xdbB = xdb['_kl']*Bu['^k_i']
L_bar = - 2*N*epsE['_kli']*Dbu['^kl'] \
        + 2*xdb['_ij']*Bub['^j'] + 2*xdbB['_li']*b['^l'] \
        + 2*A['_ik']*(Kub - DNu)['^k']
print(L_bar)
```

```
1-form on the 3-dimensional differentiable manifold Sigma
```

In [142]:
```
L_bar.display()
```

Out[142]:

$$\left(-\frac{8\left(a^5y^5 - 10\,a^3r^2y^3 + 5\,ar^4y\right)}{a^{10}y^{10} + 5\,a^8r^2y^8 + 10\,a^6r^4y^6 + 10\,a^4r^6y^4 + 5\,a^2r^8y^2 + r^{10}}\right)\mathrm{d}r + \left(\frac{8\left(5\,a^5ry^4 - 10\,a^3r^3y^2 + ar^5\right)}{a^{10}y^{10} + 5\,a^8r^2y^8 + 10\,a^6r^4y^6 + 10\,a^4r^6y^4 + 5\,a^2r^8y^2 + r^{10}}\right)\mathrm{d}y$$

In [143]:
```
S1_bar = (4*(V*Z_bar + V_bar*Z) + G*L_bar).antisymmetrize(1,2)
print(S1_bar)
```

```
Tensor field of type (0,3) on the 3-dimensional differentiable manifold
 Sigma
```

In [144]:
```
S1_bar.display()
```

Out[144]: 0

In [145]:
```
S2_bar = 4*(T_bar*V + T*V_bar) - 4*(W*Z_bar + W_bar*Z) \
        + M_bar*G - N*bd*L_bar
print(S2_bar)
```

```
Tensor field of type (0,2) on the 3-dimensional differentiable manifold
 Sigma
```

In [146]:
```
S2_bar.display()
```
Out[146]: 0

In [147]:
```
S3_bar = (4*(W*Z_bar + W_bar*Z) + N*bd*L_bar).antisymmetrize()
print(S3_bar)
```
```
2-form on the 3-dimensional differentiable manifold Sigma
```

In [148]:
```
S3_bar.display()
```
Out[148]: 0

In [149]:
```
S4_bar = 4*(T_bar*W + T*W_bar - Y*Z_bar - Y_bar*Z) \
         + M_bar*N*bd - b2*L_bar
print(S4_bar)
```
```
1-form on the 3-dimensional differentiable manifold Sigma
```

In [150]:
```
S4_bar.display()
```
Out[150]: 0

Hence all the tensors $\bar{S}^1$, $\bar{S}^2$, $\bar{S}^3$ and $\bar{S}^4$ involved in the 3+1 decomposition of the imaginary part of the Simon-Mars are zero, as they should since the Simon-Mars tensor vanishes identically for the Kerr spacetime.