Real projective plane $\mathbb{RP}^2$

This worksheet demonstrates a few capabilities of SageManifolds (version 1.0, as included in SageMath 7.5) in computations regarding the real projective plane.

Click here to download the worksheet file (ipynb format). To run it, you must start SageMath with the Jupyter notebook, via the command `sage -n jupyter`

NB: a version of SageMath at least equal to 7.5 is required to run this worksheet:

In [1]:
```
version()
```
Out[1]: 'SageMath version 7.5, Release Date: 2017-01-11'

First we set up the notebook to display mathematical objects using LaTeX rendering:

In [2]:
```
%display latex
```

We also define a viewer for 3D plots (use `'threejs'` or `'jmol'` for interactive 3D graphics):

In [3]:
```
viewer3D = 'jmol' # must be 'threejs', 'jmol', 'tachyon' or None (defau
lt)
```

Constructing the manifold

We start by declaring the real projective plane as a 2-dimensional differentiable manifold:

In [4]:
```
RP2 = Manifold(2, 'RP^2', r'\mathbb{RP}^2') ; RP2
```
Out[4]: $\mathbb{RP}^2$

Then we provide $\mathbb{RP}^2$ with some atlas. A minimal atlas on $\mathbb{RP}^2$ must have at least three charts. Such an atlas is easy to infer from the common interpretation of $\mathbb{RP}^2$ as the set of lines of $\mathbb{R}^3$ passing through the origin $(x, y, z) = (0, 0, 0)$. Let U_1 be the subset of lines that are not contained in the plane $z = 0$; this is an open set of $\mathbb{RP}^2$, so that we declare it as:

In [5]:
```
U1 = RP2.open_subset('U_1') ; U1
```
Out[5]: U_1

Any line in U_1 is uniquely determined by its intersection with the plane $z = 1$. The Cartesian coordinates $(x, y, 1)$ of the intersection point lead to an obvious coordinate system (x_1, y_1) on U_1 by setting $(x_1, y_1) = (x, y)$:

In [6]:
```
X1.<x1,y1> = U1.chart() ; X1
```
Out[6]: $(U_1, (x_1, y_1))$

Note that since we have not specified any coordinate range in the arguments of `chart()`, the range of (x_1, y_1) is $\mathbb{R}^2$.

Similarly, let U_2 be the set of lines through the origin of $\mathbb{R}^3$ that are not contained in the plane $x = 0$. Any line in U_2 is uniquely determined by its intersection $(1, y, z)$ with the plane $x = 1$, leading to coordinates $(x_2, y_2) = (y, z)$ on U_2:

In [7]:
```
U2 = RP2.open_subset('U_2')
X2.<x2,y2> = U2.chart() ; X2
```

Out[7]: $(U_2, (x_2, y_2))$

Finally, let U_3 be the set of lines through the origin of $\mathbb{R}^3$ that are not contained in the plane $y = 0$. Any line in U_3 is uniquely determined by its intersection $(x, 1, z)$ with the plane $y = 1$, leading to coordinates $(x_3, y_3) = (z, x)$ on U_3:

In [8]:
```
U3 = RP2.open_subset('U_3')
X3.<x3,y3> = U3.chart() ; X3
```

Out[8]: $(U_3, (x_3, y_3))$

We declare that the union of the three (overlapping) open domains U_1, U_2 and U_3 is $\mathbb{RP}^2$:

In [9]:
```
RP2.declare_union(U1.union(U2), U3)
U1.union(U2).union(U3)
```

Out[9]: $\mathbb{RP}^2$

At this stage, three open covers of $\mathbb{RP}^2$ have been constructed:

In [10]:
```
RP2.open_covers()
```

Out[10]: $\left[\left[\mathbb{RP}^2\right], [U_1 \cup U_2, U_3], [U_1, U_2, U_3]\right]$

Finally, to fully specify the manifold $\mathbb{RP}^2$, we give the transition maps between the various charts; the transition map between the charts X1=$(U_1, (x_1, y_1))$ and X2=$(U_2, (x_2, y_2))$ is defined on the set $U_{12} := U_1 \cap U_2$ of lines through the origin of $\mathbb{R}^3$ that are neither contained in the plane $x = 0$ ($x_1 = 0$ in U_1) nor contained in the plane $z = 0$ ($y_2 = 0$ in U_2):

In [11]:
```
X1_to_X2 = X1.transition_map(X2, (y1/x1, 1/x1), intersection_name='U_{1
2}',
                              restrictions1= x1!=0, restrictions2= y2!=0
)
X1_to_X2.display()
```

Out[11]:
$$\left\{\begin{array}{lcl} x_2 & = & \frac{y_1}{x_1} \\ y_2 & = & \frac{1}{x_1} \end{array}\right.$$

The inverse of this transition map is easily computed by Sage:

In [12]:
```
X2_to_X1 = X1_to_X2.inverse()
X2_to_X1.display()
```

Out[12]:
$$\left\{\begin{array}{lcl} x_1 & = & \frac{1}{y_2} \\ y_1 & = & \frac{x_2}{y_2} \end{array}\right.$$

The transition map between the charts X1=$(U_1,(x_1,y_1))$ and X3=$(U_3,(x_3,y_3))$ is defined on the set $U_{13} := U_1 \cap U_3$ of lines through the origin of $\mathbb{R}^3$ that are neither contained in the plane $y = 0$ ($y_1 = 0$ in U_1) nor contained in the plane $z = 0$ ($x_3 = 0$ in U_3):

In [13]:
```
X1_to_X3 = X1.transition_map(X3, (1/y1, x1/y1), intersection_name='U_{1
3}',
                                restrictions1= y1!=0, restrictions2= x3!=0
)
X1_to_X3.display()
```

Out[13]:
$$\left\{\begin{array}{lcl} x_3 & = & \frac{1}{y_1} \\ y_3 & = & \frac{x_1}{y_1} \end{array}\right.$$

In [14]:
```
X3_to_X1 = X1_to_X3.inverse()
X3_to_X1.display()
```

Out[14]:
$$\left\{\begin{array}{lcl} x_1 & = & \frac{y_3}{x_3} \\ y_1 & = & \frac{1}{x_3} \end{array}\right.$$

Finally, the transition map between the charts X2=$(U_2,(x_2,y_2))$ and X3=$(U_3,(x_3,y_3))$ is defined on the set $U_{23} := U_2 \cap U_3$ of lines through the origin of $\mathbb{R}^3$ that are neither contained in the plane $y = 0$ ($x_2 = 0$ in U_2) nor contained in the plane $x = 0$ ($y_3 = 0$ in U_3):

In [15]:
```
X2_to_X3 = X2.transition_map(X3, (y2/x2, 1/x2), intersection_name='U_{2
3}',
                                restrictions1= x2!=0, restrictions2= y3!=0
)
X2_to_X3.display()
```

Out[15]:
$$\left\{\begin{array}{lcl} x_3 & = & \frac{y_2}{x_2} \\ y_3 & = & \frac{1}{x_2} \end{array}\right.$$

In [16]:
```
X3_to_X2 = X2_to_X3.inverse()
X3_to_X2.display()
```

Out[16]:
$$\left\{\begin{array}{lcl} x_2 & = & \frac{1}{y_3} \\ y_2 & = & \frac{x_3}{y_3} \end{array}\right.$$

At this stage, the manifold $\mathbb{RP}^2$ is fully constructed. It has been provided with the following atlas:

In [17]:
```
RP2.atlas()
```

Out[17]:
$$\left[\left(U_1,(x_1,y_1)\right), \left(U_2,(x_2,y_2)\right), \left(U_3,(x_3,y_3)\right), \left(U_{12},(x_1,y_1)\right), \left(U_{12},(x_2,y_2)\right), \left(U_{13},(x_1,y_1)\right), \left(U_{13},(x_3,y_3)\right), \left(U_{23},(x_2,y_2)\right), \left(U_{23},(x_3,y_3)\right)\right]$$

Note that, in addition to the three chart we have defined, the atlas comprises subcharts on the intersection domains U_{12}, U_{13} and U_{23}. These charts can be obtained by the method `restrict()`:

In [18]:
```
U12 = U1.intersection(U2)
U13 = U1.intersection(U3)
U23 = U2.intersection(U3)
X1.restrict(U12)
```

Out[18]: $(U_{12}, (x_1, y_1))$

In [19]:
```
X1.restrict(U12) is RP2.atlas()[3]
```

Out[19]: True

Non-orientability of $\mathbb{RP}^2$

It is well known that $\mathbb{RP}^2$ is not an orientable manifold. To illustrate this, let us make an attempt to construct a global non-vanishing 2-form ϵ on $\mathbb{RP}^2$. If we succeed, this would provide a volume form and $\mathbb{RP}^2$ would be orientable. We start by declaring ϵ as a 2-form on $\mathbb{RP}^2$:

In [20]:
```
eps = RP2.diff_form(2, name='eps', latex_name=r'\epsilon')
print(eps)
```

```
2-form eps on the 2-dimensional differentiable manifold RP^2
```

We set the value of ϵ on domain U_1 to be $\mathrm{d}x_1 \wedge \mathrm{d}y_1$ by demanding that the component ϵ_{01} of ϵ with respect to coordinates (x_1, y_1) is one, as follows:

In [21]:
```
e1 = X1.frame() ; e1
```

Out[21]: $\left(U_1, \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}\right)\right)$

In [22]:
```
eps[e1,0,1] = 1
eps.display(e1)
```

Out[22]: $\epsilon = \mathrm{d}x_1 \wedge \mathrm{d}y_1$

If we ask for the expression of ϵ in terms of the coframe $(\mathrm{d}x_2, \mathrm{d}y_2)$ associated with the chart X2 on $U_{12} = U_1 \cap U_2$, we get

In [23]:
```
eps.display(X2.frame().restrict(U12), chart=X2.restrict(U12))
```

Out[23]: $\epsilon = \frac{1}{y_2^3} \mathrm{d}x_2 \wedge \mathrm{d}y_2$

Now, the complement of U_{12} in U_2 is defined by $y_2 = 0$. The above expression shows that it is not possible to extend smoothly ϵ to the whole domain U_2. We conclude that starting from $\mathrm{d}x_1 \wedge \mathrm{d}y_1$ on U_1, it is not possible to get a regular non-vanishing 2-form on $\mathbb{RP}^2$. This of course follows from the fact that $\mathbb{RP}^2$ is not orientable.

Steiner map (Roman surface)

Let us first define $\mathbb{R}^3$ as a 3-dimensional manifold, with a single-chart atlas (Cartesian coordinates Y):

In [24]:
```
R3 = Manifold(3, 'R^3', r'\mathbb{R}^3')
Y.<x,y,z> = R3.chart()
```

The Steiner map is a map $\mathbb{RP}^2 \to \mathbb{R}^3$ defined as follows:

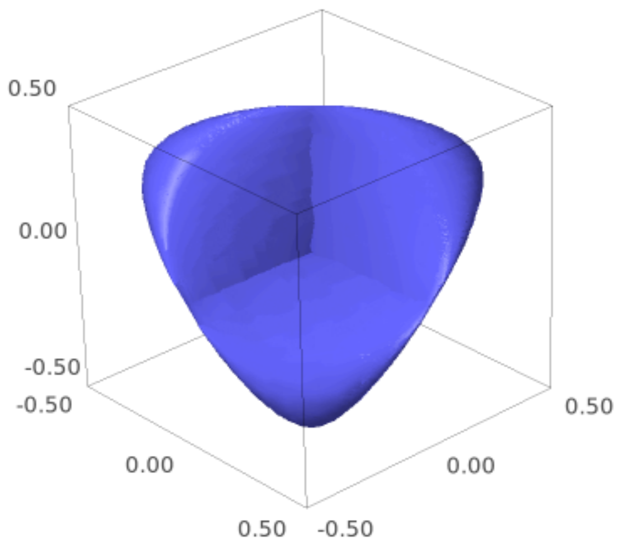
In [25]:
```
Phi = RP2.diff_map(R3, {(X1,Y): [y1/(1+x1^2+y1^2), x1/(1+x1^2+y1^2), x1
*y1/(1+x1^2+y1^2)],
                        (X2,Y): [x2*y2/(1+x2^2+y2^2), y2/(1+x2^2+y2^2),
 x2/(1+x2^2+y2^2)],
                        (X3,Y): [x3/(1+x3^2+y3^2), x3*y3/(1+x3^2+y3^2),
 y3/(1+x3^2+y3^2)]},
                   name='Phi', latex_name=r'\Phi')
Phi.display()
```

Out[25]:

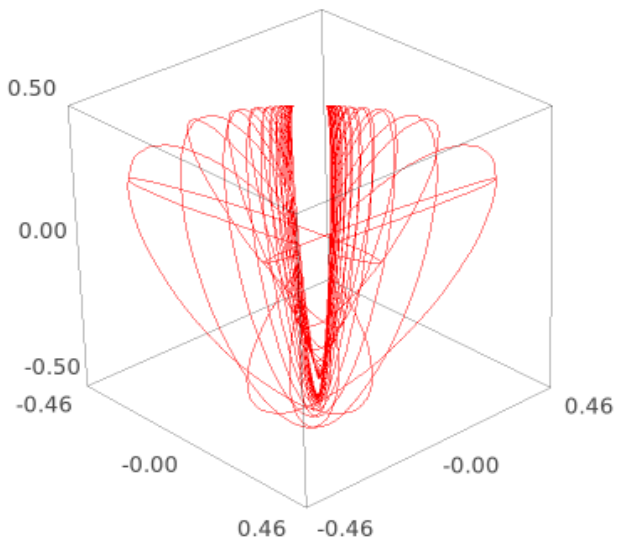
$$\begin{array}{llcl} \Phi: & \mathbb{RP}^2 & \longrightarrow & \mathbb{R}^3 \\ \text{on } U_1: & (x_1, y_1) & \longmapsto & (x, y, z) = \left(\frac{y_1}{x_1^2+y_1^2+1}, \frac{x_1}{x_1^2+y_1^2+1}, \frac{x_1 y_1}{x_1^2+y_1^2+1} \right) \\ \text{on } U_2: & (x_2, y_2) & \longmapsto & (x, y, z) = \left(\frac{x_2 y_2}{x_2^2+y_2^2+1}, \frac{y_2}{x_2^2+y_2^2+1}, \frac{x_2}{x_2^2+y_2^2+1} \right) \\ \text{on } U_3: & (x_3, y_3) & \longmapsto & (x, y, z) = \left(\frac{x_3}{x_3^2+y_3^2+1}, \frac{x_3 y_3}{x_3^2+y_3^2+1}, \frac{y_3}{x_3^2+y_3^2+1} \right) \end{array}$$

Φ is a topological immersion of $\mathbb{RP}^2$ into $\mathbb{R}^3$, but it is not a smooth immersion (contrary to the Apéry map below): its differential is not injective at $(x_1, y_1) = (0, 1)$ and $(x_1, y_1) = (1, 0)$. The image of Φ is a self-intersecting surface of $\mathbb{R}^3$, called the **Roman surface**:

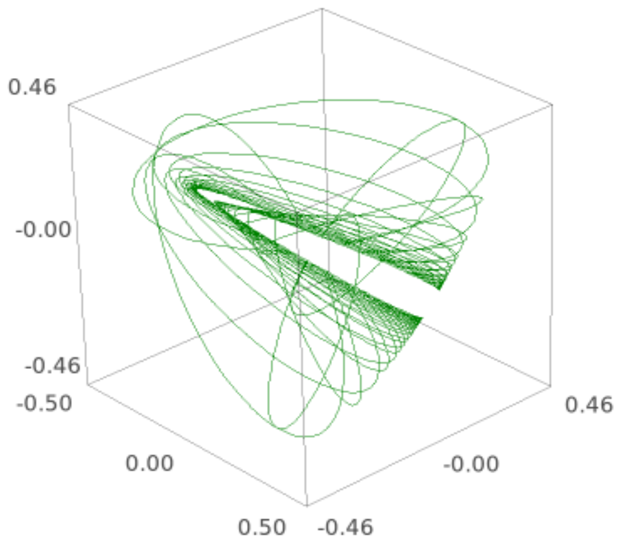
In [26]:
```
g1 = parametric_plot3d(Phi.expr(X1,Y), (x1,-10,10), (y1,-10,10), plot_p
oints=[100,100])
g2 = parametric_plot3d(Phi.expr(X2,Y), (x2,-10,10), (y2,-10,10), plot_p
oints=[100,100])
g3 = parametric_plot3d(Phi.expr(X3,Y), (x3,-10,10), (y3,-10,10), plot_p
oints=[100,100])
show(g1+g2+g3, viewer=viewer3D)
```



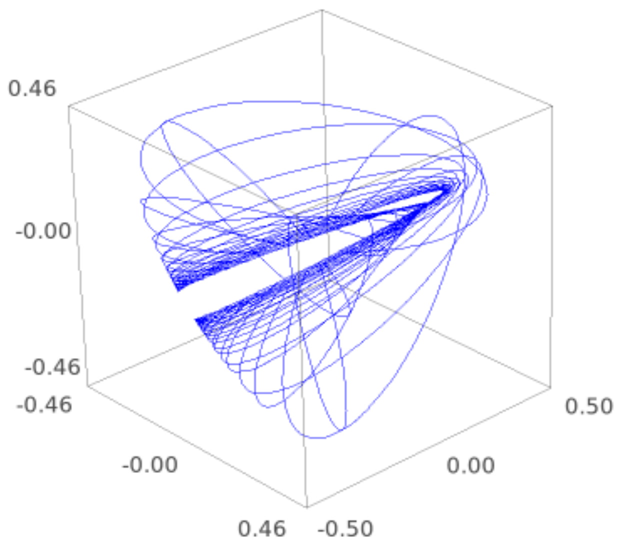
In [27]:
```
gX1 = X1.plot(Y, mapping=Phi, max_range=16, number_values=24, plot_poin
ts=100,
              label_axes=False)
gX2 = X2.plot(Y, mapping=Phi, max_range=16, number_values=24, plot_poin
ts=100,
              label_axes=False, color='green')
gX3 = X3.plot(Y, mapping=Phi, max_range=16, number_values=24, plot_poin
ts=100,
              label_axes=False, color='blue')
show(gX1, viewer=viewer3D)
```



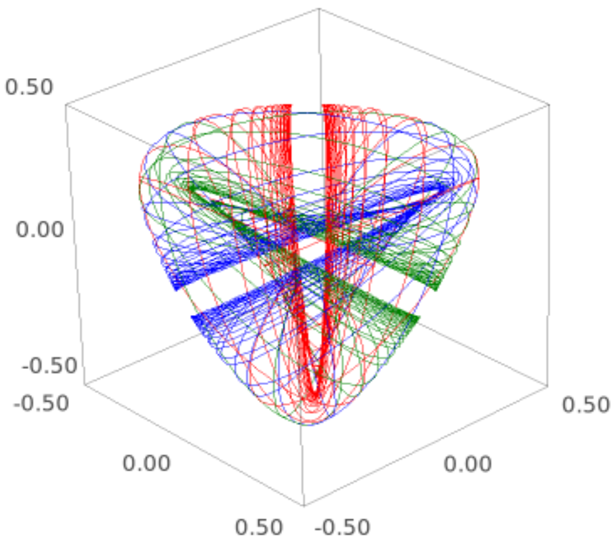
In [28]:
```
show(gX2, viewer=viewer3D)
```



In [29]:
```
show(gX3, viewer=viewer3D)
```



In [30]:
```
show(gX1+gX2+gX3, viewer=viewer3D)
```



Apéry map (Boy surface)

The Apéry map [Apéry, Adv. Math. **61**, 185 (1986)] is a smooth immersion $\Psi : \mathbb{RP}^2 \to \mathbb{R}^3$. In terms of the charts X1, X2, X3 introduced above, it is defined as follows:

In [31]:
```
fx = ((2*x^2-y^2-z^2)*(x^2+y^2+z^2)+2*y*z*(y^2-z^2)+z*x*(x^2-z^2)+x*y*(y^2-z^2))/2 ; fx
```

Out[31]:

$$\frac{1}{2}\left(y^2 - z^2\right)xy + \frac{1}{2}\left(x^2 - z^2\right)xz + \left(y^2 - z^2\right)yz + \frac{1}{2}\left(2\,x^2 - y^2 - z^2\right)\left(x^2 + y^2 + z^2\right)$$

In [32]:
```
fy = sqrt(3)/2*((y^2-z^2)*(x^2+y^2+z^2)+z*x*(z^2-x^2)+x*y*(y^2-x^2)) ; fy
```

Out[32]:

$$-\frac{1}{2}\sqrt{3}\left(\left(x^2 - y^2\right)xy + \left(x^2 - z^2\right)xz - \left(x^2 + y^2 + z^2\right)\left(y^2 - z^2\right)\right)$$

In [33]:
```
fz = (x+y+z)*((x+y+z)^3/4+(y-x)*(z-y)*(x-z)) ; fz
```

Out[33]:

$$\frac{1}{4}\left((x + y + z)^3 + 4\,(x - y)(x - z)(y - z)\right)(x + y + z)$$

In [34]:
```
a = sqrt(1+x1^2+y1^2)
fx1 = fx.subs(x=x1/a, y=y1/a, z=1/a).simplify_full()
fy1 = fy.subs(x=x1/a, y=y1/a, z=1/a).simplify_full()
fz1 = fz.subs(x=x1/a, y=y1/a, z=1/a).simplify_full()
```

In [35]:
```
a = sqrt(1+x2^2+y2^2)
fx2 = fx.subs(x=1/a, y=x2/a, z=y2/a).simplify_full()
fy2 = fy.subs(x=1/a, y=x2/a, z=y2/a).simplify_full()
fz2 = fz.subs(x=1/a, y=x2/a, z=y2/a).simplify_full()
```

In [36]:
```
a = sqrt(1+x3^2+y3^2)
fx3 = fx.subs(x=y3/a, y=1/a, z=x3/a).simplify_full()
fy3 = fy.subs(x=y3/a, y=1/a, z=x3/a).simplify_full()
fz3 = fz.subs(x=y3/a, y=1/a, z=x3/a).simplify_full()
```

In [37]:
```
Psi = RP2.diff_map(R3, {(X1,Y): [fx1, fy1, fz1], (X2,Y): [fx2, fy2, fz2
],
                        (X3,Y): [fx3, fy3, fz3]}, name='Psi', latex_nam
e=r'\Psi')
Psi.display()
```

Out[37]:

$$\Psi: \mathbb{RP}^2 \longrightarrow \mathbb{R}^3$$

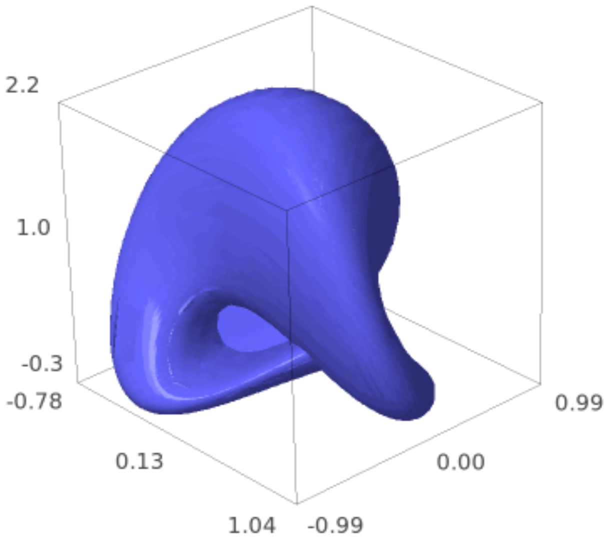
$$\text{on } U_1: (x_1, y_1) \longmapsto (x, y, z) = \left(\frac{2\,x_1^4+(x_1+2)y_1^3-y_1^4+x_1^3+\left(x_1^2-2\right)y_1^2+x_1^2-(x_1+2)y_1-x_1-1}{2\,\left(x_1^4+y_1^4+2\,\left(x_1^2+1\right)y_1^2+2\,x_1^2+1\right)}, -\frac{\sqrt{3}x_1^3y_1-\sqrt{3}x_1^2y_1^2-\sqrt{3}x_1y_1^3-\sqrt{3}y_1^4+\sqrt{3}x_1^3+\sqrt{3}x_1^2-\sqrt{3}x_1+\sqrt{3}}{2\,\left(x_1^4+y_1^4+2\,\left(x_1^2+1\right)y_1^2+2\,x_1^2+1\right)}, \frac{x_1^4+y_1^4+6\,\left(x_1^2+2\,x_1+1\right)y_1^2+8\,y_1^3+6\,x_1^2+4\,\left(2\,x_1^3+3\,x_1^2+3\,x_1\right)y_1+8\,x_1+1}{4\,\left(x_1^4+y_1^4+2\,\left(x_1^2+1\right)y_1^2+2\,x_1^2+1\right)} \right)$$

$$\text{on } U_2: (x_2, y_2) \longmapsto (x, y, z) = \left(-\frac{x_2^4+(2\,x_2+1)y_2^3+y_2^4-x_2^3+\left(2\,x_2^2+x_2-1\right)y_2^2-x_2^2-\left(2\,x_2^3+1\right)y_2-2}{2\,\left(x_2^4+y_2^4+2\,\left(x_2^2+1\right)y_2^2+2\,x_2^2+1\right)}, \frac{\sqrt{3}x_2^4-\sqrt{3}y_2^4+\sqrt{3}x_2^3+\sqrt{3}y_2^3+\sqrt{3}x_2^2-\sqrt{3}y_2^2-\sqrt{3}x_2-\sqrt{3}y_2}{2\,\left(x_2^4+y_2^4+2\,\left(x_2^2+1\right)y_2^2+2\,x_2^2+1\right)}, \frac{x_2^4+y_2^4+6\,\left(x_2^2+2\,x_2+1\right)y_2^2+8\,y_2^3+6\,x_2^2+4\,\left(2\,x_2^3+3\,x_2^2+3\,x_2\right)y_2+8\,x_2+1}{4\,\left(x_2^4+y_2^4+2\,\left(x_2^2+1\right)y_2^2+2\,x_2^2+1\right)} \right)$$

$$\text{on } U_3: (x_3, y_3) \longmapsto (x, y, z) = \left(-\frac{x_3^4-x_3y_3^3-2\,y_3^4+2\,x_3^3-\left(x_3^2+1\right)y_3^2+2\,x_3^2+\left(x_3^3+x_3^2-1\right)y_3-2\,x_3+1}{2\,\left(x_3^4+y_3^4+2\,\left(x_3^2+1\right)y_3^2+2\,x_3^2+1\right)}, -\frac{\sqrt{3}x_3^4+\left(\sqrt{3}x_3+\sqrt{3}\right)y_3^3+\left(\sqrt{3}x_3^2-\sqrt{3}\right)y_3^2-\left(\sqrt{3}x_3^3+\sqrt{3}\right)y_3-\sqrt{3}}{2\,\left(x_3^4+y_3^4+2\,\left(x_3^2+1\right)y_3^2+2\,x_3^2+1\right)}, \frac{x_3^4+y_3^4+6\,\left(x_3^2+2\,x_3+1\right)y_3^2+8\,y_3^3+6\,x_3^2+4\,\left(2\,x_3^3+3\,x_3^2+3\,x_3\right)y_3+8\,x_3+1}{4\,\left(x_3^4+y_3^4+2\,\left(x_3^2+1\right)y_3^2+2\,x_3^2+1\right)} \right)$$

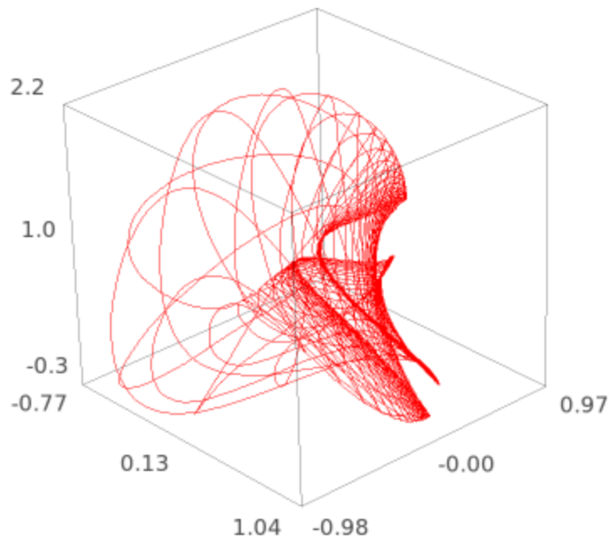
The image of Ψ is a self-intersecting surface of $\mathbb{R}^3$, called the **Boy surface,** after Werner Boy (1879-1914):

In [38]:
```
g1 = parametric_plot3d(Psi.expr(X1,Y), (x1,-10,10), (y1,-10,10), plot_p
oints=[100,100])
g2 = parametric_plot3d(Psi.expr(X2,Y), (x2,-10,10), (y2,-10,10), plot_p
oints=[100,100])
g3 = parametric_plot3d(Psi.expr(X3,Y), (x3,-10,10), (y3,-10,10), plot_p
oints=[100,100])
show(g1+g2+g3, viewer=viewer3D)
```

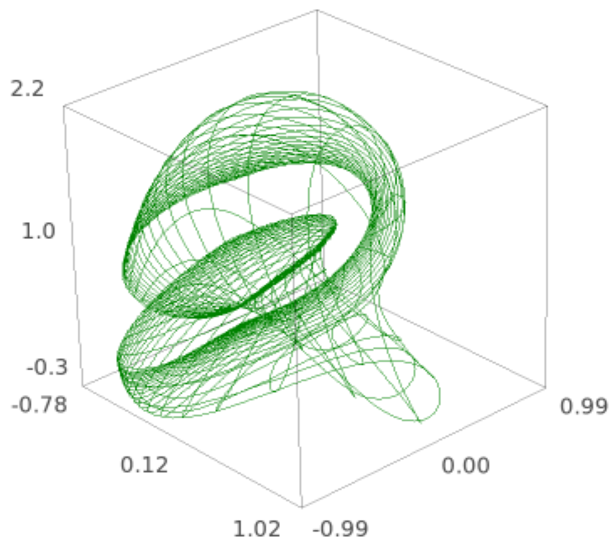


In [39]:
```
gX1 = X1.plot(Y, mapping=Psi, number_values=40, plot_points=100, label_
axes=False)
gX2 = X2.plot(Y, mapping=Psi, number_values=40, plot_points=100, label_
axes=False,
              color='green')
gX3 = X3.plot(Y, mapping=Psi, number_values=40, plot_points=100, label_
axes=False,
              color='blue')
```

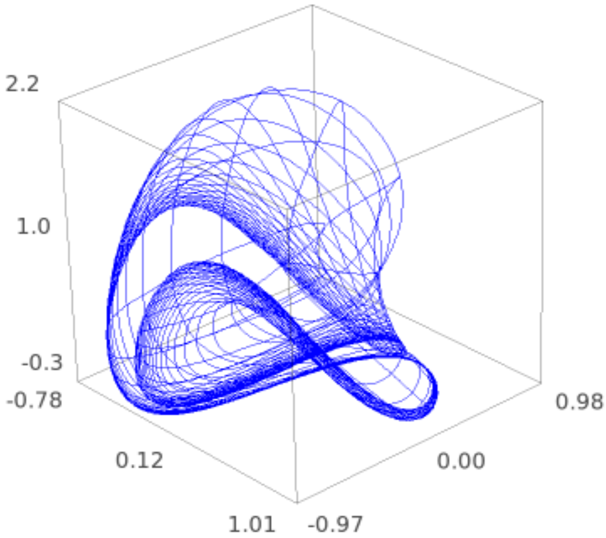
In [40]:
```
show(gX1, viewer=viewer3D)
```



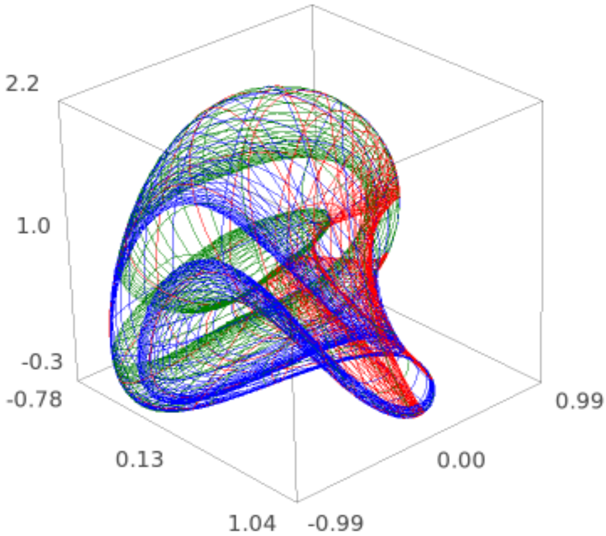
In [41]:
```
show(gX2, viewer=viewer3D)
```



In [42]:
```
show(gX3, viewer=viewer3D)
```



In [43]:
```
show(gX1+gX2+gX3, viewer=viewer3D)
```



In [ ]:
```
```