Carter-Penrose diagram of Schwarzschild spacetime

This worksheet demonstrates a few capabilities of SageManifolds (version 1.0, as included in SageMath 7.5) in computations regarding the Carter-Penrose diagram of Schwarzschild spacetime. It is used to illustrate the lectures Geometry and physics of black holes

Click here to download the worksheet file (ipynb format). To run it, you must start SageMath with the Jupyter notebook, via the command `sage -n jupyter`

NB: a version of SageMath at least equal to 7.5 is required to run this worksheet:

In [1]:
```
version()
```

Out[1]: 'SageMath version 7.5, Release Date: 2017-01-11'

First we set up the notebook to display mathematical objects using LaTeX formatting:

In [2]:
```
%display latex
```

Spacetime

We declare the spacetime manifold M:

In [3]:
```
M = Manifold(4, 'M')
print(M)
```

```
4-dimensional differentiable manifold M
```

The Schwarzschild-Droste domain

The domain of Schwarzschild-Droste coordinates is $M_{\rm SD} = M_{\rm I} \cup M_{\rm II}$:

In [4]:
```
M_SD = M.open_subset('M_SD', latex_name=r'M_{\rm SD}')
M_I = M_SD.open_subset('M_I', latex_name=r'M_{\rm I}')
M_II = M_SD.open_subset('M_II', latex_name=r'M_{\rm II}')
M_SD.declare_union(M_I, M_II)
```

The Schwarzschild-Droste coordinates (t, r, θ, ϕ):

In [5]:
```
X_SD.<t,r,th,ph> = M_SD.chart(r't r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi
):\phi')
m = var('m', domain='real') ; assume(m>=0)
X_SD.add_restrictions(r!=2*m)
X_SD
```

Out[5]: $(M_{\rm SD}, (t, r, \theta, \phi))$

In [6]:
```
X_SD_I = X_SD.restrict(M_I, r>2*m) ; X_SD_I
```

Out[6]: $(M_{\rm I}, (t, r, \theta, \phi))$

In [7]:
```
X_SD_II = X_SD.restrict(M_II, r<2*m) ; X_SD_II
```

Out[7]: $(M_{\mathrm{II}}, (t, r, \theta, \phi))$

In [8]:
```
M.default_chart()
```

Out[8]: $(M_{\mathrm{SD}}, (t, r, \theta, \phi))$

In [9]:
```
M.atlas()
```

Out[9]: $[(M_{\mathrm{SD}}, (t, r, \theta, \phi)), (M_{\mathrm{I}}, (t, r, \theta, \phi)), (M_{\mathrm{II}}, (t, r, \theta, \phi))]$

Kruskal-Szekeres coordinates

In [10]:
```
X_KS.<T,X,th,ph> = M.chart(r'T X th:(0,pi):\theta ph:(0,2*pi):\phi')
X_KS.add_restrictions(T^2 < 1 + X^2)
X_KS
```

Out[10]: $(M, (T, X, \theta, \phi))$

In [11]:
```
X_KS_I = X_KS.restrict(M_I, [X>0, T<X, T>-X]) ; X_KS_I
```

Out[11]: $(M_{\mathrm{I}}, (T, X, \theta, \phi))$

In [12]:
```
X_KS_II = X_KS.restrict(M_II, [T>0, T>abs(X)]) ; X_KS_II
```

Out[12]: $(M_{\mathrm{II}}, (T, X, \theta, \phi))$

In [13]:
```
SD_I_to_KS = X_SD_I.transition_map(X_KS_I, [sqrt(r/(2*m)-1)*exp(r/(4*m)
)*sinh(t/(4*m)),
                                            sqrt(r/(2*m)-1)*exp(r/(4*m)
)*cosh(t/(4*m)),
                                            th, ph])
SD_I_to_KS.display()
```

Out[13]:
$$\left\{\begin{array}{lcl} T & = & \sqrt{\frac{r}{2\,m} - 1}e^{\left(\frac{r}{4\,m}\right)} \sinh\left(\frac{t}{4\,m}\right) \\ X & = & \sqrt{\frac{r}{2\,m} - 1} \cosh\left(\frac{t}{4\,m}\right)e^{\left(\frac{r}{4\,m}\right)} \\ \theta & = & \theta \\ \phi & = & \phi \end{array}\right.$$

In [14]:
```
SD_II_to_KS = X_SD_II.transition_map(X_KS_II, [sqrt(1-r/(2*m))*exp(r/(4
*m))*cosh(t/(4*m)),
                                                sqrt(1-r/(2*m))*exp(r/(4
*m))*sinh(t/(4*m)),
                                                th, ph])
SD_II_to_KS.display()
```

Out[14]:
$$\left\{\begin{array}{lcl} T & = & \sqrt{-\frac{r}{2\,m} + 1} \cosh\left(\frac{t}{4\,m}\right)e^{\left(\frac{r}{4\,m}\right)} \\ X & = & \sqrt{-\frac{r}{2\,m} + 1}e^{\left(\frac{r}{4\,m}\right)} \sinh\left(\frac{t}{4\,m}\right) \\ \theta & = & \theta \\ \phi & = & \phi \end{array}\right.$$

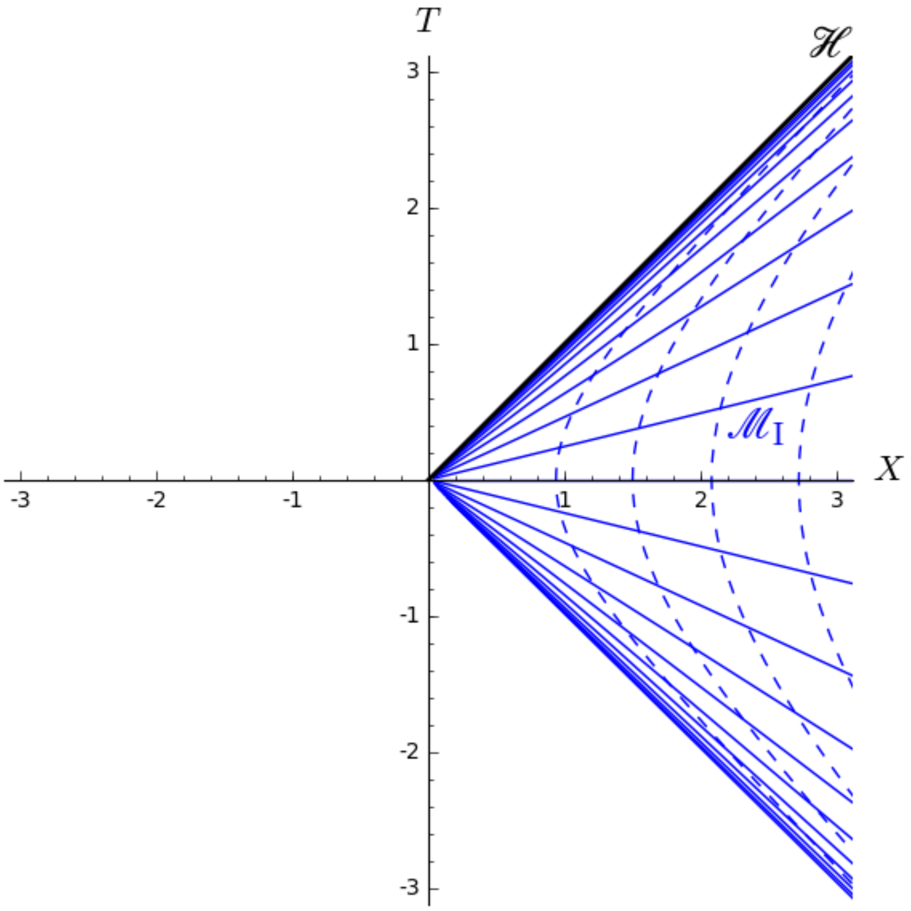
Plot of Schwarzschild-Droste grid on M_{I} in terms of KS coordinates

In [15]:
```
graph = X_SD_I.plot(X_KS, ambient_coords=(X,T), fixed_coords={th:pi/2,p
h:pi},
                    ranges={t:(-10,10), r:(2.001,5)}, steps={t:1, r:0.5
},
                    style={t:'--', r:'-'}, color='blue', parameters={m:
1})
```

Adding the Schwarzschild horizon to the plot:

In [16]:
```
hor = line([(0,0), (4,4)], color='black', thickness=2) \
      + text(r'$\mathscr{H}$', (3, 2.7), fontsize=20, color='black')
```

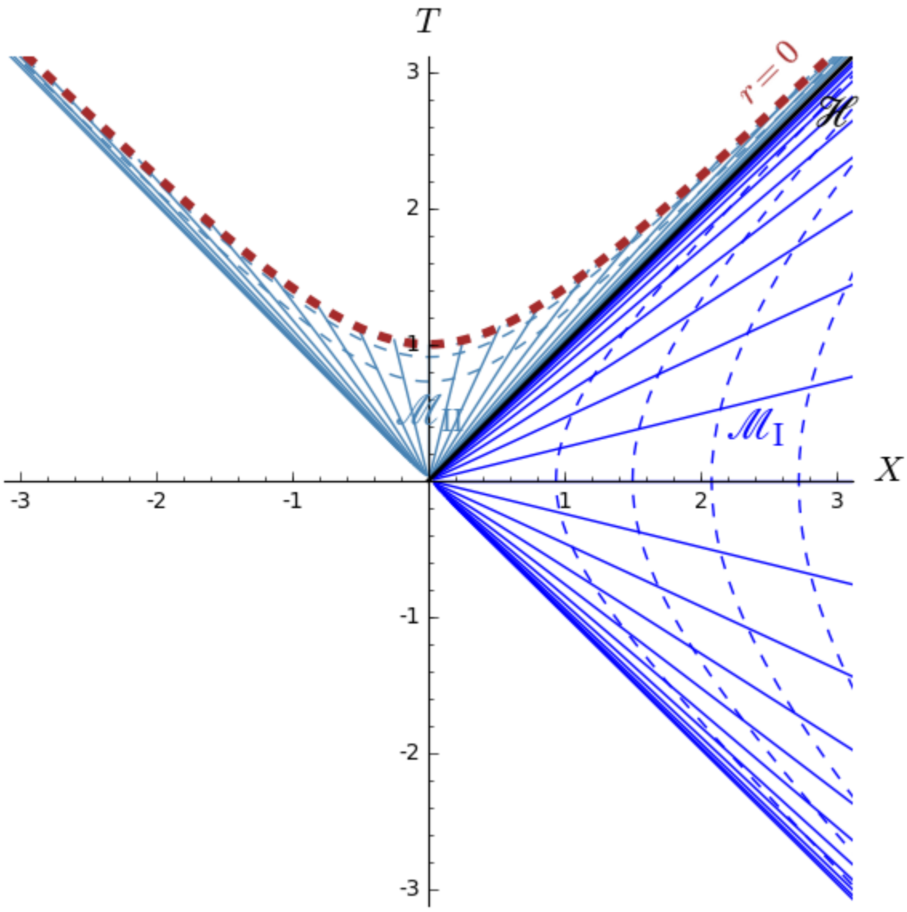
In [17]:
```
hor2 = line([(0,0), (4,4)], color='black', thickness=2) \
      + text(r'$\mathscr{H}$', (2.95, 3.2), fontsize=20, color='black')
region_labels = text(r'$\mathscr{M}_{\rm I}$', (2.4, 0.4), fontsize=20,
 color='blue')
graph2 = graph + hor2 + region_labels
show(graph2, xmin=-3, xmax=3, ymin=-3, ymax=3)
```



Adding the curvature singularity $r = 0$ to the plot:

In [18]:
```
sing = X_SD_II.plot(X_KS, fixed_coords={r:0, th:pi/2, ph:pi}, ambient_c
oords=(X,T),
                    color='brown', thickness=4, style='--', parameters=
{m:1}) \
       + text(r'$r=0$', (2.5, 3), rotation=45, fontsize=16, color='brow
n')
```

In [19]:
```
graph += X_SD_II.plot(X_KS, ambient_coords=(X,T), fixed_coords={th:pi/2
,ph:pi},
                      ranges={t:(-10,10), r:(0.001,1.999)}, steps={t:1,
 r:0.5},
                      style={t:'--', r:'-'}, color='steelblue', paramet
ers={m:1})
region_labels = text(r'$\mathscr{M}_{\rm I}$', (2.4, 0.4), fontsize=20,
 color='blue') + \
                text(r'$\mathscr{M}_{\rm II}$', (0, 0.5), fontsize=20,
color='steelblue')
graph += hor + sing + region_labels
show(graph, xmin=-3, xmax=3, ymin=-3, ymax=3)
```



Extension to $M_{\rm III}$ and $M_{\rm IV}$

In [20]:
```
M_III = M.open_subset('M_III', latex_name=r'M_{\rm III}', coord_def={X_
KS: [X<0, X<T, T<-X]})
X_KS_III = X_KS.restrict(M_III) ; X_KS_III
```

Out[20]: $(M_{\rm III}, (T, X, \theta, \phi))$

In [21]:
```
M_IV = M.open_subset('M_IV', latex_name=r'M_{\rm IV}', coord_def={X_KS:
 [T<0, T<-abs(X)]})
X_KS_IV = X_KS.restrict(M_IV) ; X_KS_IV
```

Out[21]: $(M_{\rm IV}, (T, X, \theta, \phi))$

Schwarzschild-Droste coordinates in $M_{\rm III}$ and $M_{\rm IV}$:

In [22]:
```
X_SD_III.<t,r,th,ph> = M_III.chart(r't r:(2*m,+oo) th:(0,pi):\theta ph:
(0,2*pi):\phi')
X_SD_III
```

Out[22]: $(M_{\mathrm{III}}, (t, r, \theta, \phi))$

In [23]:
```
SD_III_to_KS = X_SD_III.transition_map(X_KS_III, [-sqrt(r/(2*m)-1)*exp(
r/(4*m))*sinh(t/(4*m)),
                                                    - sqrt(r/(2*m)-1)*exp
(r/(4*m))*cosh(t/(4*m)),
                                                    th, ph])
SD_III_to_KS.display()
```

Out[23]:
$$\begin{cases} T = -\sqrt{\frac{r}{2\,m} - 1}\, e^{\left(\frac{r}{4\,m}\right)} \sinh\left(\frac{t}{4\,m}\right) \\ X = -\sqrt{\frac{r}{2\,m} - 1}\, \cosh\left(\frac{t}{4\,m}\right) e^{\left(\frac{r}{4\,m}\right)} \\ \theta = \theta \\ \phi = \phi \end{cases}$$

In [24]:
```
X_SD_IV.<t,r,th,ph> = M_IV.chart(r't r:(0,2*m) th:(0,pi):\theta ph:(0,2
*pi):\phi')
X_SD_IV
```

Out[24]: $(M_{\mathrm{IV}}, (t, r, \theta, \phi))$

In [25]:
```
SD_IV_to_KS = X_SD_IV.transition_map(X_KS_IV, [-sqrt(1-r/(2*m))*exp(r/(
4*m))*cosh(t/(4*m)),
                                                 -sqrt(1-r/(2*m))*exp(r/(
4*m))*sinh(t/(4*m)),
                                                  th, ph])
SD_IV_to_KS.display()
```

Out[25]:
$$\begin{cases} T = -\sqrt{-\frac{r}{2\,m} + 1}\, \cosh\left(\frac{t}{4\,m}\right) e^{\left(\frac{r}{4\,m}\right)} \\ X = -\sqrt{-\frac{r}{2\,m} + 1}\, e^{\left(\frac{r}{4\,m}\right)} \sinh\left(\frac{t}{4\,m}\right) \\ \theta = \theta \\ \phi = \phi \end{cases}$$

Standard compactified coordinates

The coordinates $(\hat{T}, \hat{X}, \theta, \varphi)$ associated with the conformal compactification of the Schwarzschild spacetime are

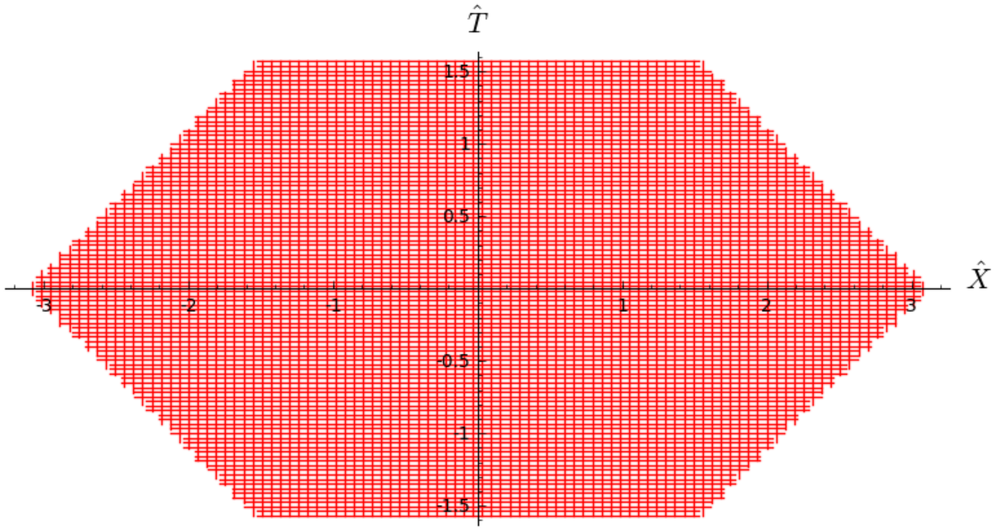
In [26]:
```
X_C.<T1,X1,th,ph> = M.chart(r'T1:(-pi/2,pi/2):\hat{T} X1:(-pi,pi):\hat{
X} th:(0,pi):\theta ph:(0,2*pi):\varphi')
X_C.add_restrictions([-pi+abs(X1)<T1, T1<pi-abs(X1)])
X_C
```

Out[26]: $\left(M, (\hat{T}, \hat{X}, \theta, \varphi)\right)$

The chart of compactified coordinates plotted in terms of itself:

In [27]:
```
X_C.plot(X_C, ambient_coords=(X1,T1), number_values=100)
```

Out[27]:



The transition map from Kruskal-Szekeres coordinates to the compactified ones:

In [28]:
```
KS_to_C = X_KS.transition_map(X_C, [atan(T+X)+atan(T-X),
                                    atan(T+X)-atan(T-X),
                                    th, ph])
print(KS_to_C)
KS_to_C.display()
```

```
Change of coordinates from Chart (M, (T, X, th, ph)) to Chart (M, (T1,
X1, th, ph))
```

Out[28]:

$$\begin{cases} \hat{T} & = & \arctan(T+X) + \arctan(T-X) \\ \hat{X} & = & \arctan(T+X) - \arctan(T-X) \\ \theta & = & \theta \\ \varphi & = & \varphi \end{cases}$$

Transition map between the Schwarzschild-Droste chart and the chart of compactified coordinates

The transition map is obtained by composition of previously defined ones:

In [29]:
```
SD_I_to_C = KS_to_C.restrict(M_I) * SD_I_to_KS
print(SD_I_to_C)
SD_I_to_C.display()
```

```
Change of coordinates from Chart (M_I, (t, r, th, ph)) to Chart (M_I, (
T1, X1, th, ph))
```

Out[29]:

$$\left\{\begin{array}{lcl} \hat{T} & = & \arctan\left(\frac{\left(\sqrt{2}\cosh\left(\frac{t}{4\,m}\right)e^{\left(\frac{r}{4\,m}\right)}+\sqrt{2}e^{\left(\frac{r}{4\,m}\right)}\sinh\left(\frac{t}{4\,m}\right)\right)\sqrt{-2\,m+r}}{2\,\sqrt{m}}\right)+\arctan\left(-\frac{\left(\sqrt{2}\cosh\left(\frac{t}{4\,m}\right)e^{\left(\frac{r}{4\,m}\right)}-\sqrt{2}e^{\left(\frac{r}{4\,m}\right)}\sinh\left(\frac{t}{4\,m}\right)\right)\sqrt{-2\,m+r}}{2\,\sqrt{m}}\right) \\ \hat{X} & = & \arctan\left(\frac{\left(\sqrt{2}\cosh\left(\frac{t}{4\,m}\right)e^{\left(\frac{r}{4\,m}\right)}+\sqrt{2}e^{\left(\frac{r}{4\,m}\right)}\sinh\left(\frac{t}{4\,m}\right)\right)\sqrt{-2\,m+r}}{2\,\sqrt{m}}\right)-\arctan\left(-\frac{\left(\sqrt{2}\cosh\left(\frac{t}{4\,m}\right)e^{\left(\frac{r}{4\,m}\right)}-\sqrt{2}e^{\left(\frac{r}{4\,m}\right)}\sinh\left(\frac{t}{4\,m}\right)\right)\sqrt{-2\,m+r}}{2\,\sqrt{m}}\right) \\ \theta & = & \theta \\ \varphi & = & \varphi \end{array}\right.$$

In [30]:
```
SD_II_to_C = KS_to_C.restrict(M_II) * SD_II_to_KS
print(SD_II_to_C)
SD_II_to_C.display()
```

```
Change of coordinates from Chart (M_II, (t, r, th, ph)) to Chart (M_II,
 (T1, X1, th, ph))
```

Out[30]:

$$\left\{\begin{array}{lcl} \hat{T} & = & \arctan\left(\frac{\left(\sqrt{2}\cosh\left(\frac{t}{4\,m}\right)e^{\left(\frac{r}{4\,m}\right)}+\sqrt{2}e^{\left(\frac{r}{4\,m}\right)}\sinh\left(\frac{t}{4\,m}\right)\right)\sqrt{2\,m-r}}{2\,\sqrt{m}}\right)-\arctan\left(-\frac{\left(\sqrt{2}\cosh\left(\frac{t}{4\,m}\right)e^{\left(\frac{r}{4\,m}\right)}-\sqrt{2}e^{\left(\frac{r}{4\,m}\right)}\sinh\left(\frac{t}{4\,m}\right)\right)\sqrt{2\,m-r}}{2\,\sqrt{m}}\right) \\ \hat{X} & = & \arctan\left(\frac{\left(\sqrt{2}\cosh\left(\frac{t}{4\,m}\right)e^{\left(\frac{r}{4\,m}\right)}+\sqrt{2}e^{\left(\frac{r}{4\,m}\right)}\sinh\left(\frac{t}{4\,m}\right)\right)\sqrt{2\,m-r}}{2\,\sqrt{m}}\right)+\arctan\left(-\frac{\left(\sqrt{2}\cosh\left(\frac{t}{4\,m}\right)e^{\left(\frac{r}{4\,m}\right)}-\sqrt{2}e^{\left(\frac{r}{4\,m}\right)}\sinh\left(\frac{t}{4\,m}\right)\right)\sqrt{2\,m-r}}{2\,\sqrt{m}}\right) \\ \theta & = & \theta \\ \varphi & = & \varphi \end{array}\right.$$

In [31]:
```
SD_III_to_C = KS_to_C.restrict(M_III) * SD_III_to_KS
print(SD_III_to_C)
SD_III_to_C.display()
```

```
Change of coordinates from Chart (M_III, (t, r, th, ph)) to Chart (M_II
I, (T1, X1, th, ph))
```

Out[31]:

$$\left\{\begin{array}{lcl} \hat{T} & = & -\arctan\left(\frac{\left(\sqrt{2}\cosh\left(\frac{t}{4\,m}\right)e^{\left(\frac{r}{4\,m}\right)}+\sqrt{2}e^{\left(\frac{r}{4\,m}\right)}\sinh\left(\frac{t}{4\,m}\right)\right)\sqrt{-2\,m+r}}{2\,\sqrt{m}}\right) - \arctan\left(-\frac{\left(\sqrt{2}\cosh\left(\frac{t}{4\,m}\right)e^{\left(\frac{r}{4\,m}\right)}-\sqrt{2}e^{\left(\frac{r}{4\,m}\right)}\sinh\left(\frac{t}{4\,m}\right)\right)\sqrt{-2\,m+r}}{2\,\sqrt{m}}\right) \\ \hat{X} & = & -\arctan\left(\frac{\left(\sqrt{2}\cosh\left(\frac{t}{4\,m}\right)e^{\left(\frac{r}{4\,m}\right)}+\sqrt{2}e^{\left(\frac{r}{4\,m}\right)}\sinh\left(\frac{t}{4\,m}\right)\right)\sqrt{-2\,m+r}}{2\,\sqrt{m}}\right) + \arctan\left(-\frac{\left(\sqrt{2}\cosh\left(\frac{t}{4\,m}\right)e^{\left(\frac{r}{4\,m}\right)}-\sqrt{2}e^{\left(\frac{r}{4\,m}\right)}\sinh\left(\frac{t}{4\,m}\right)\right)\sqrt{-2\,m+r}}{2\,\sqrt{m}}\right) \\ \theta & = & \theta \\ \varphi & = & \varphi \end{array}\right.$$

In [32]:
```
SD_IV_to_C = KS_to_C.restrict(M_IV) * SD_IV_to_KS
print(SD_IV_to_C)
SD_IV_to_C.display()
```

```
Change of coordinates from Chart (M_IV, (t, r, th, ph)) to Chart (M_IV,
 (T1, X1, th, ph))
```

Out[32]:

$$\left\{\begin{array}{lcl} \hat{T} & = & -\arctan\left(\frac{\left(\sqrt{2}\cosh\left(\frac{t}{4\,m}\right)e^{\left(\frac{r}{4\,m}\right)}+\sqrt{2}e^{\left(\frac{r}{4\,m}\right)}\sinh\left(\frac{t}{4\,m}\right)\right)\sqrt{2\,m-r}}{2\,\sqrt{m}}\right) + \arctan\left(-\frac{\left(\sqrt{2}\cosh\left(\frac{t}{4\,m}\right)e^{\left(\frac{r}{4\,m}\right)}-\sqrt{2}e^{\left(\frac{r}{4\,m}\right)}\sinh\left(\frac{t}{4\,m}\right)\right)\sqrt{2\,m-r}}{2\,\sqrt{m}}\right) \\ \hat{X} & = & -\arctan\left(\frac{\left(\sqrt{2}\cosh\left(\frac{t}{4\,m}\right)e^{\left(\frac{r}{4\,m}\right)}+\sqrt{2}e^{\left(\frac{r}{4\,m}\right)}\sinh\left(\frac{t}{4\,m}\right)\right)\sqrt{2\,m-r}}{2\,\sqrt{m}}\right) - \arctan\left(-\frac{\left(\sqrt{2}\cosh\left(\frac{t}{4\,m}\right)e^{\left(\frac{r}{4\,m}\right)}-\sqrt{2}e^{\left(\frac{r}{4\,m}\right)}\sinh\left(\frac{t}{4\,m}\right)\right)\sqrt{2\,m-r}}{2\,\sqrt{m}}\right) \\ \theta & = & \theta \\ \varphi & = & \varphi \end{array}\right.$$

Carter-Penrose diagram

The diagram is obtained by plotting the curves of constant Schwarzschild-Droste coordinates with respect to the compactified chart.

In [33]:
```
r_tab = [2.01*m, 2.1*m, 2.5*m, 4*m, 8*m, 12*m, 20*m, 100*m]
curves_t = dict()
for r0 in r_tab:
    curves_t[r0] = M.curve({X_SD_I: [t, r0, pi/2, pi]}, (t,-oo,+oo))
    curves_t[r0].coord_expr(X_C.restrict(M_I))
```

In [34]:
```
graph_t = Graphics()
for r0 in r_tab:
    graph_t += curves_t[r0].plot(X_C, ambient_coords=(X1,T1), prange=(-
150, -10),
                                 parameters={m:1}, plot_points=100, col
or='blue', style='--')
    graph_t += curves_t[r0].plot(X_C, ambient_coords=(X1,T1), prange=(-
10, 10),
                                 parameters={m:1}, plot_points=100, col
or='blue', style='--')
    graph_t += curves_t[r0].plot(X_C, ambient_coords=(X1,T1), prange=(1
0, 150),
                                 parameters={m:1}, plot_points=100, col
or='blue', style='--')
```

In [35]:
```
t_tab = [-50*m, -20*m, -10*m, -5*m, -2*m, 0, 2*m, 5*m, 10*m, 20*m, 50*m
]
curves_r = dict()
for t0 in t_tab:
    curves_r[t0] = M.curve({X_SD_I: [t0, r, pi/2, pi]}, (r, 2*m, +oo))
    curves_r[t0].coord_expr(X_C.restrict(M_I))
```

In [36]:
```
graph_r = Graphics()
for t0 in t_tab:
    graph_r += curves_r[t0].plot(X_C, ambient_coords=(X1,T1), prange=(2
.0001, 4),
                                 parameters={m:1}, plot_points=100, col
or='blue')
    graph_r += curves_r[t0].plot(X_C, ambient_coords=(X1,T1), prange=(4
, 1000),
                                 parameters={m:1}, plot_points=100, col
or='blue')
```

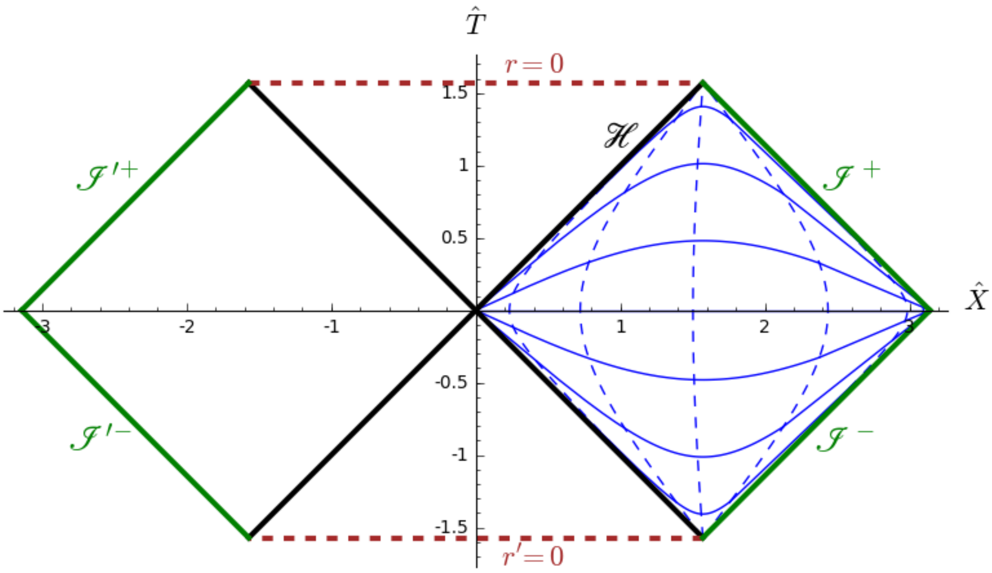
In [37]:
```
bifhor = line([(-pi/2,-pi/2), (pi/2,pi/2)], color='black', thickness=3)
 + \
         line([(-pi/2,pi/2), (pi/2,-pi/2)], color='black', thickness=3)
 + \
         text(r'$\mathscr{H}$', (1, 1.2), fontsize=20, color='black')
```

In [38]:
```
sing1 = X_SD_II.plot(X_C, fixed_coords={r:0, th:pi/2, ph:pi}, ambient_c
oords=(X1,T1),
                    max_range=200, number_values=30, color='brown', thi
ckness=3,
                    style='--', parameters={m:1}) + \
        text(r'$r=0$', (0.4, 1.7), fontsize=16, color='brown')
sing2 = X_SD_IV.plot(X_C, fixed_coords={r:0, th:pi/2, ph:pi}, ambient_c
oords=(X1,T1),
                    max_range=200, number_values=30, color='brown', thi
ckness=3,
                    style='--', parameters={m:1}) + \
        text(r"$r'=0$", (0.4, -1.7), fontsize=16, color='brown')
sing = sing1 + sing2
```

In [39]:
```
scri = line([(pi,0), (pi/2,pi/2)], color='green', thickness=3) + \
       text(r"$\mathscr{I}^+$", (2.6, 0.9), fontsize=20, color='green')
 + \
       line([(pi/2, -pi/2), (pi,0)], color='green', thickness=3) + \
       text(r"$\mathscr{I}^-$", (2.55, -0.9), fontsize=20, color='green
') + \
       line([(-pi,0), (-pi/2,pi/2)], color='green', thickness=3) + \
       text(r"${\mathscr{I}'}^+$", (-2.55, 0.9), fontsize=20, color='gr
een') + \
       line([(-pi/2, -pi/2), (-pi,0)], color='green', thickness=3) + \
       text(r"${\mathscr{I}'}^-$", (-2.6, -0.9), fontsize=20, color='gr
een')
```

In [40]:
```
region_labels = text(r'$\mathscr{M}_{\rm I}$', (2, 0.4), fontsize=20, c
olor='blue',
                     background_color='white') + \
                text(r'$\mathscr{M}_{\rm II}$', (0.4, 1), fontsize=20,
color='steelblue',
                     background_color='white') + \
                text(r'$\mathscr{M}_{\rm III}$', (-2, 0.4), fontsize=20
, color='chocolate',
                     background_color='white') + \
                text(r'$\mathscr{M}_{\rm IV}$', (0.4, -1), fontsize=20,
 color='gold',
                     background_color='white')
```

In [41]:
```
graph = graph_t + graph_r
show(graph + bifhor + sing + scri, aspect_ratio=1)
```



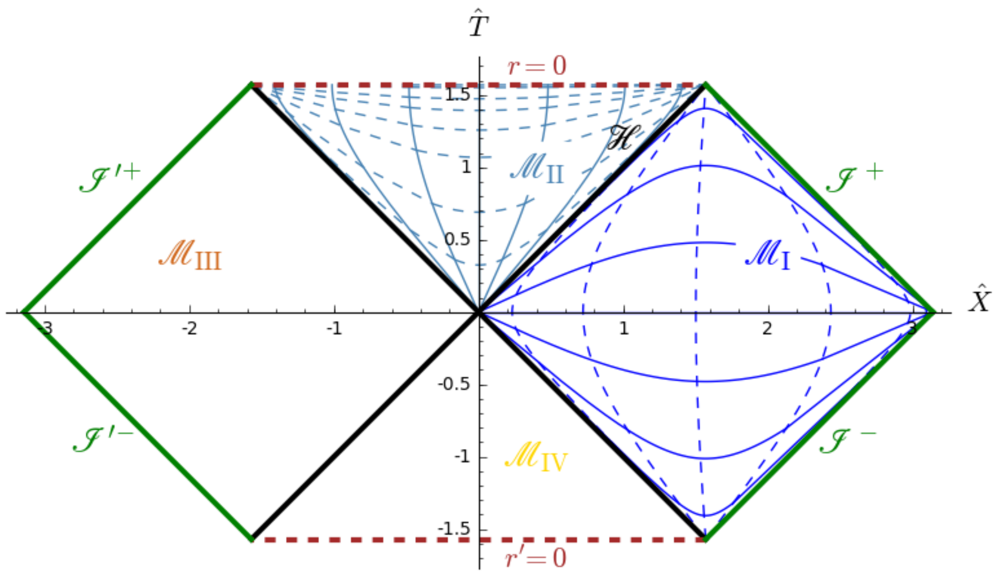
In [42]:
```
r_tab = [0.1*m, 0.5*m, m, 1.25*m, 1.5*m, 1.7*m, 1.9*m, 1.98*m]
curves_t = dict()
for r0 in r_tab:
    curves_t[r0] = M.curve({X_SD_II: [t, r0, pi/2, pi]}, (t,-oo,+oo))
    curves_t[r0].coord_expr(X_C.restrict(M_II))
```

In [43]:
```
graph_t = Graphics()
for r0 in r_tab:
    graph_t += curves_t[r0].plot(X_C, ambient_coords=(X1,T1), prange=(-
150, -2),
                                 parameters={m:1}, plot_points=50, colo
r='steelblue', style='--')
    graph_t += curves_t[r0].plot(X_C, ambient_coords=(X1,T1), prange=(-
2, 2),
                                 parameters={m:1}, plot_points=50, colo
r='steelblue', style='--')
    graph_t += curves_t[r0].plot(X_C, ambient_coords=(X1,T1), prange=(2
, 150),
                                 parameters={m:1}, plot_points=50, colo
r='steelblue', style='--')
```

In [44]:
```
t_tab = [-20*m, -10*m, -5*m, -2*m, 0, 2*m, 5*m, 10*m, 20*m]
curves_r = dict()
for t0 in t_tab:
    curves_r[t0] = M.curve({X_SD_II: [t0, r, pi/2, pi]}, (r, 0, 2*m))
    curves_r[t0].coord_expr(X_C.restrict(M_II))
```

In [45]:
```
graph_r = Graphics()
for t0 in t_tab:
    graph_r += curves_r[t0].plot(X_C, ambient_coords=(X1,T1), prange=(0
.001, 1.9999),
                                 parameters={m:1}, plot_points=100, col
or='steelblue')
```

In [46]:
```
graph += graph_t + graph_r
show(graph + bifhor + sing + scri + region_labels, aspect_ratio=1)
```



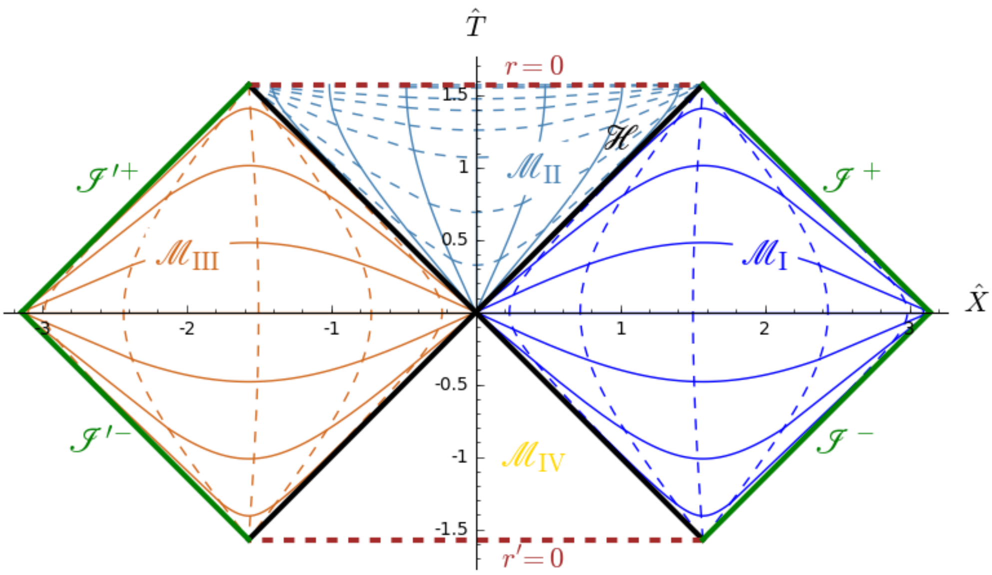
In [47]:
```
r_tab = [2.01*m, 2.1*m, 2.5*m, 4*m, 8*m, 12*m, 20*m, 100*m]
curves_t = dict()
for r0 in r_tab:
    curves_t[r0] = M.curve({X_SD_III: [t, r0, pi/2, pi]}, (t,-oo,+oo))
    curves_t[r0].coord_expr(X_C.restrict(M_III))
```

In [48]:
```
graph_t = Graphics()
for r0 in r_tab:
    graph_t += curves_t[r0].plot(X_C, ambient_coords=(X1,T1), prange=(-
150, -10),
                                 parameters={m:1}, plot_points=100, col
or='chocolate', style='--')
    graph_t += curves_t[r0].plot(X_C, ambient_coords=(X1,T1), prange=(-
10, 10),
                                 parameters={m:1}, plot_points=100, col
or='chocolate', style='--')
    graph_t += curves_t[r0].plot(X_C, ambient_coords=(X1,T1), prange=(1
0, 150),
                                 parameters={m:1}, plot_points=100, col
or='chocolate', style='--')
```

In [49]:
```
t_tab = [-50*m, -20*m, -10*m, -5*m, -2*m, 0, 2*m, 5*m, 10*m, 20*m, 50*m
]
curves_r = dict()
for t0 in t_tab:
    curves_r[t0] = M.curve({X_SD_III: [t0, r, pi/2, pi]}, (r, 2*m, +oo)
)
    curves_r[t0].coord_expr(X_C.restrict(M_III))
```

In [50]:
```
graph_r = Graphics()
for t0 in t_tab:
    graph_r += curves_r[t0].plot(X_C, ambient_coords=(X1,T1), prange=(2
.0001, 4),
                                 parameters={m:1}, plot_points=100, col
or='chocolate')
    graph_r += curves_r[t0].plot(X_C, ambient_coords=(X1,T1), prange=(4
, 1000),
                                 parameters={m:1}, plot_points=100, col
or='chocolate')
```

In [51]:
```
graph += graph_t + graph_r
show(graph + bifhor + sing + scri + region_labels, aspect_ratio=1)
```



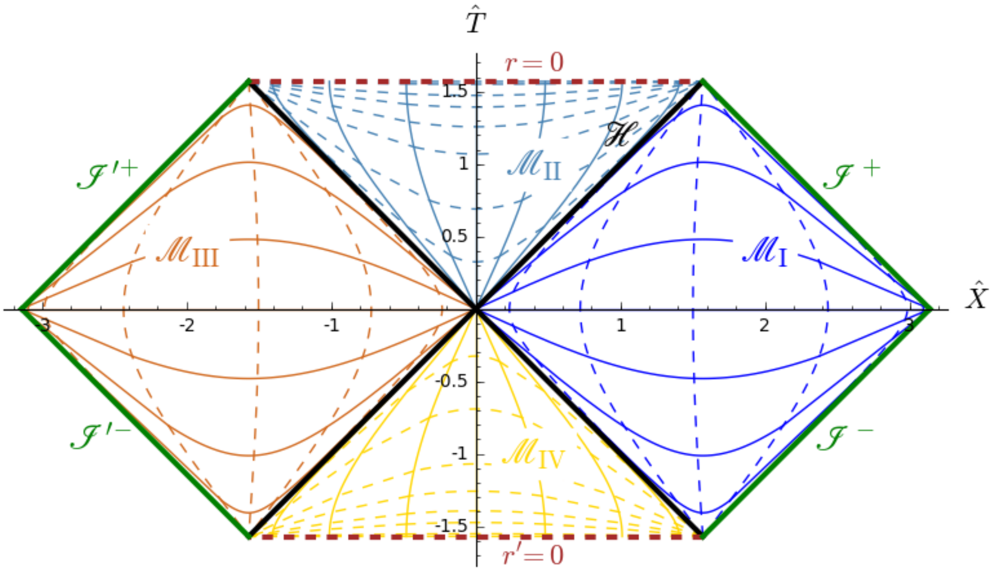
In [52]:
```
r_tab = [0.1*m, 0.5*m, m, 1.25*m, 1.5*m, 1.7*m, 1.9*m, 1.98*m]
curves_t = dict()
for r0 in r_tab:
    curves_t[r0] = M.curve({X_SD_IV: [t, r0, pi/2, pi]}, (t,-oo,+oo))
    curves_t[r0].coord_expr(X_C.restrict(M_IV))
```

In [53]:
```
graph_t = Graphics()
for r0 in r_tab:
    graph_t += curves_t[r0].plot(X_C, ambient_coords=(X1,T1), prange=(-
150, -2),
                                 parameters={m:1}, plot_points=50, colo
r='gold', style='--')
    graph_t += curves_t[r0].plot(X_C, ambient_coords=(X1,T1), prange=(-
2, 2),
                                 parameters={m:1}, plot_points=50, colo
r='gold', style='--')
    graph_t += curves_t[r0].plot(X_C, ambient_coords=(X1,T1), prange=(2
, 150),
                                 parameters={m:1}, plot_points=50, colo
r='gold', style='--')
```

In [54]:
```
t_tab = [-20*m, -10*m, -5*m, -2*m, 0, 2*m, 5*m, 10*m, 20*m]
curves_r = dict()
for t0 in t_tab:
    curves_r[t0] = M.curve({X_SD_IV: [t0, r, pi/2, pi]}, (r, 0, 2*m))
    curves_r[t0].coord_expr(X_C.restrict(M_IV))
```

In [55]:
```
graph_r = Graphics()
for t0 in t_tab:
    graph_r += curves_r[t0].plot(X_C, ambient_coords=(X1,T1), prange=(0
.001, 1.9999),
                                 parameters={m:1}, plot_points=100, col
or='gold')
```

In [56]:
```
graph += graph_t + graph_r
graph += bifhor + sing + scri + region_labels
show(graph, aspect_ratio=1)
```



In [57]:
```
graph.save('max_carter-penrose-std.pdf', aspect_ratio=1)
```