# 3+1 Simon-Mars tensor in the $\delta = 2$ Tomimatsu-Sato spacetime

This worksheet demonstrates a few capabilities of SageManifolds (version 1.0, as included in SageMath 7.5) in computations regarding the 3+1 decomposition of the Simon-Mars tensor in the $\delta = 2$ Tomimatsu-Sato spacetime. The results obtained here are used in the article arXiv:1412.6542.

Click here to download the worksheet file (ipynb format). To run it, you must start SageMath with the Jupyter notebook, via the command `sage -n jupyter`

*NB:* a version of SageMath at least equal to 7.5 is required to run this worksheet:

In [1]:
```
version()
```

Out[1]: 'SageMath version 7.5.1, Release Date: 2017-01-15'

First we set up the notebook to display mathematical objects using LaTeX rendering:

In [2]:
```
%display latex
```

Since some computations are quite long, we ask for running them in parallel on 8 cores:

In [3]:
```
Parallelism().set(nproc=8)
```

## Tomimatsu-Sato spacetime

The Tomimatsu-Sato metric is an exact stationary and axisymmetric solution of the vacuum Einstein equation, which is asymptotically flat and has a non-zero angular momentum. It has been found in 1972 by A. Tomimatsu and H. Sato [Phys. Rev. Lett. **29**, 1344 (1972)], as a solution of the Ernst equation. It is actually the member $\delta = 2$ of a larger family of solutions parametrized by a positive integer $\delta$ and exhibited by Tomimatsu and Sato in 1973 [Prog. Theor. Phys. **50**, 95 (1973)], the member $\delta = 1$ being nothing but the Kerr metric. We refer to [Manko, Prog. Theor. Phys. **127**, 1057 (2012)] for a discussion of the properties of this solution.

## Spacelike hypersurface

We consider some hypersurface $\Sigma$ of a spacelike foliation $(\Sigma_t)_{t\in\mathbb{R}}$ of $\delta = 2$ Tomimatsu-Sato spacetime; we declare $\Sigma_t$ as a 3-dimensional manifold:

In [4]:
```
Sig = Manifold(3, 'Sigma', r'\Sigma', start_index=1)
```

On $\Sigma$, we consider the prolate spheroidal coordinates $(x, y, \phi)$, with $x \in (1, +\infty)$, $y \in (-1, 1)$ and $\phi \in (0, 2\pi)$ :

In [5]:
```
X.<r,y,ph> = Sig.chart(r'x:(1,+oo) y:(-1,1) ph:(0,2*pi):\phi')
print X ; X
```

```
Chart (Sigma, (x, y, ph))
```

Out[5]: $(\Sigma, (x, y, \phi))$

## Riemannian metric on $\Sigma$

The Tomimatsu-Sato metric depens on three parameters: the integer $\delta$, the real number $p \in [0, 1]$, and the total mass $m$:

In [6]:
```
var('d, p, m')
assume(m>0)
assumptions()
```

Out[6]:

$$[\mathtt{x\ is\ real}, x > 1, \mathtt{y\ is\ real}, y > (-1), y < 1, \mathtt{ph\ is\ real}, \phi > 0, \phi < 2\pi, m > 0]$$

We set $\delta = 2$ and choose a specific value for $p$, namely $p = 1/5$:

In [7]:
```
d = 2
p = 1/5
```

Furthermore, without any loss of generality, we may set $m = 1$ (this simply fixes some length scale):

In [8]:
```
m=1
```

The parameter $q$ is related to $p$ by $p^2 + q^2 = 1$:

In [9]:
```
q = sqrt(1-p^2)
```

Some shortcut notations:

In [10]:
```
AA2 = (p^2*(x^2-1)^2+q^2*(1-y^2)^2)^2 \
      - 4*p^2*q^2*(x^2-1)*(1-y^2)*(x^2-y^2)^2
BB2 = (p^2*x^4+2*p*x^3-2*p*x+q^2*y^4-1)^2 \
      + 4*q^2*y^2*(p*x^3-p*x*y^2-y^2+1)^2
CC2 = p^3*x*(1-x^2)*(2*(x^4-1)+(x^2+3)*(1-y^2)) \
      + p^2*(x^2-1)*((x^2-1)*(1-y^2)-4*x^2*(x^2-y^2)) \
      + q^2*(1-y^2)^3*(p*x+1)
```

The Riemannian metric $\gamma$ induced by the spacetime metric $g$ on $\Sigma$:

In [11]:
```
gam = Sig.riemannian_metric('gam', latex_name=r'\gamma')
gam[1,1] = m^2*BB2/(p^2*d^2*(x^2-1)*(x^2-y^2)^3)
gam[2,2] = m^2*BB2/(p^2*d^2*(y^2-1)*(-x^2+y^2)^3)
gam[3,3] = - m^2*(y^2-1)*(p^2*BB2^2*(x^2-1)
                          + 4*q^2*d^2*CC2^2*(y^2-1))/(AA2*BB2*d^2)
gam.display()
```

Out[11]:

$$\gamma = \left( \frac{96 \left(x^3 - xy^2 - 5\,y^2 + 5\right)^2 y^2 + \left(x^4 + 24\,y^4 \cdots\right.}{100 \left(x^2 - y^2\right)^3 \left(x^2 - 1 \cdots\right.} \right.$$

$$\otimes \mathrm{d}x + \left( -\frac{96 \left(x^3 - xy^2 - 5\,y^2 + 5\right)^2 y^2 + \left(x^4 + 24\,y \cdots\right.}{100 \left(x^2 - y^2\right)^3 \left(y^2 - \cdots\right.} \right.$$

$$\otimes \mathrm{d}y$$

$$\left( \left( 96 \left(x^3 - xy^2 - 5\,y^2 + 5\right)^2 y^2 + \left(x^4 + 24\,y^4 + 10\,x^3 - 10 \cdots \right. \right. \right.$$

$$+ 9600$$

$$-\frac{\left(24 \left(y^2 - 1\right)^3 (x+5) + \left(2\,x^4 - \left(x^2 + 3\right)\left(y^2 - 1\right) - 2\right)\left(x^2 \cdots \right. \right. \quad \left(4 \left(x^2 - y^2\right)x^2 + \left(x^2 - 1\right)\left(y^2 - 1\right)\right)\left(x^2 - 1\right)\Big)}{100 \left(96 \left(x^2 - y^2\right)^2 \left(x^2 - 1\right)\left(y^2 - 1\right) + \left(\left(x^2 - 1\right)^2 + 24 \left(y^2 - 1\right)^2\right)^2\right)\left(96 \left(x^3 - xy^2 \cdots\right.\right.}$$

A view of the non-vanishing components of $\gamma$ w.r.t. coordinates $(x, y, \phi)$:

In [12]:
```
gam.display_comp()
```

Out[12]:

$$\gamma_{x\,x} = \frac{96 \left(x^3 - xy^2 - 5\,y^2 + 5\right)^2 y^2 + \left(x^4 + 24\,y^4 + 10\,x^3 - 10\,x - 25\right)^2}{100 \left(x^2 - y^2\right)^3 \left(x^2 - 1\right)}$$

$$\gamma_{y\,y} = -\frac{96 \left(x^3 - xy^2 - 5\,y^2 + 5\right)^2 y^2 + \left(x^4 + 24\,y^4 + 10\,x^3 - 10\,x - 25\right)^2}{100 \left(x^2 - y^2\right)^3 \left(y^2 - 1\right)}$$

$$\gamma_{\phi\,\phi} = -\frac{\left(\left(96 \left(x^3 - xy^2 - 5\,y^2 + 5\right)^2 y^2 + \left(x^4 + 24\,y^4 + 10\,x^3 - 10\,x - 25\right)^2\right)^2 \left(x^2 - 1\right) + 9600 \quad \left(24 \left(y^2 - 1\right)^3 (x+5) + \left(2\,x^4 - \left(x^2 + 3\right)\left(y^2 - 1\right) - 2\right)\left(x^2 - 1\right)x + 5 \left(4 \left(x^2 - y^2\right)x^2 + \left(x^2 - 1\right)\left(y^2 - 1\right)\right)\left(x^2 - 1\right)\right)}{100 \left(96 \left(x^2 - y^2\right)^2 \left(x^2 - 1\right)\left(y^2 - 1\right) + \left(\left(x^2 - 1\right)^2 + 24 \left(y^2 - 1\right)^2\right)^2\right)\left(96 \left(x^3 - xy^2 - 5\,y^2 + 5\right)^2 y^2 + \left(x^4 + 24\,y^4 + 1 \cdots\right.\right.}$$

The expression of the metric determinant with respect to the default chart (coordinates $(x, y, \phi)$):

In [13]:
```
gam.determinant().expr()
```

Out[13]:

$$\frac{\begin{aligned}&x^{18} + 60\,x^{17} + 331776\,\left(x^{2} - 1\right)y^{16} + 1599\,x^{16} + 25880\,x^{15} + 110592\,\left(x^{4} + 15\,x^{3} + 99\,x^{2} + 485\,x + 1200\right)y^{14} + 266700\,x^{14} + 1555560\,x^{13} \\ &- 9216\,\left(17\,x^{6} + 60\,x^{5} - 417\,x^{4} - 3040\,x^{3} - 13425\,x^{2} - 31020\,x - 16975\right)y^{12} + 3533300\,x^{12} - 4005000\,x^{11} \\ &+ 9216\,\left(9\,x^{8} - 60\,x^{7} - 509\,x^{6} - 2430\,x^{5} - 9525\,x^{4} - 24260\,x^{3} - 71775\,x^{2} - 227250\,x - 290600\right)y^{10} - 17787450\,x^{10} - 18420000\,x^{9} \\ &+ 5760\,\left(7\,x^{10} + 90\,x^{9} + 473\,x^{8} + 2460\,x^{7} + 10050\,x^{6} + 15200\,x^{5} + 53790\,x^{4} + 120900\,x^{3} + 198455\,x^{2} + 741350\,x + 1103625\right)y^{8} \\ &+ 15656250\,x^{8} + 31485000\,x^{7} \\ &- 192\,\left(143\,x^{12} + 675\,x^{11} - 1043\,x^{10} - 7575\,x^{9} - 52650\,x^{8} - 224850\,x^{7} - 156150\,x^{6} + 1001250\,x^{5} + 3726075\,x^{4} + 6217375\,x^{3} + 4145625\,x^{2} + 19413125\,x + 33330000\right)y^{6} \\ &+ 3527500\,x^{6} + 12975000\,x^{5} \\ &+ 96\,\left(93\,x^{14} - 105\,x^{13} - 1693\,x^{12} - 13470\,x^{11} - 99575\,x^{10} - 222675\,x^{9} - 149025\,x^{8} - 1024500\,x^{7} - 2270025\,x^{6} + 2366625\,x^{5} + 9545625\,x^{4} + 11931250\,x^{3} + 451875\,x^{2} + 11346875\,x + 28273125\right)y^{4} \\ &+ 80032500\,x^{4} + 102025000\,x^{3} \\ &+ 192\,\left(x^{16} + 30\,x^{15} + 399\,x^{14} + 3955\,x^{13} + 19950\,x^{12} + 3765\,x^{11} + 19850\,x^{10} + 197000\,x^{9} + 47025\,x^{8} + 77000\,x^{7} + 646875\,x^{6} - 598125\,x^{5} - 2642500\,x^{4} - 2896875\,x^{3} + 1117500\,x^{2} + 1581250\,x - 687500\right)y^{2} \\ &- 78609375\,x^{2} - 180937500\,x - 150390625\end{aligned}}{1000000\,\left(x^{14} + \left(x^{2} - 1\right)y^{12} - x^{12} - 6\,\left(x^{4} - x^{2}\right)y^{10} + 15\,\left(x^{6} - x^{4}\right)y^{8} - 20\,\left(x^{8} - x^{6}\right)y^{6} + 15\,\left(x^{10} - x^{8}\right)y^{4} - 6\,\left(x^{12} - x^{10}\right)y^{2}\right)}$$

### Lapse function and shift vector

In [14]:
```
N2 = AA2/BB2 - 2*m*q*CC2*(y^2-1)/BB2*(2*m*q*CC2*(y^2-1)
               /(BB2*(m^2*(y^2-1)*(p^2*BB2^2*(x^2-1)
               +4*q^2*d^2*CC2^2*(y^2-1))/(AA2*BB2*d^2))))
N2.simplify_full()
```

Out[14]:

$$\frac{x^{10} + 20\,x^{9} + 576\,\left(x^{2} - 1\right)y^{8} + 99\,x^{8} - 40\,x^{7} + 96\,\left(x^{4} + 10\,x^{3} + 24\,x^{2} - 10\,x - 25\right)y^{6} - 350\,x^{6} - 480\,x^{5} - 48\,\left(3\,x^{6} + 10\,x^{5} - 3\,x^{4} + 20\,x^{3} + 125\,x^{2} - 30\,x - 125\right)y^{4} + 350\,x^{4} + 1000\,x^{3} + 96\,\left(x^{8} - x^{6} + 10\,x^{5} - 10\,x^{3} + 25\,x^{2} - 25\right)y^{2} + 525\,x^{2} - 500\,x - 625}{x^{10} + 40\,x^{9} + 576\,\left(x^{2} - 1\right)y^{8} + 699\,x^{8} + 7920\,x^{7} + 96\,\left(x^{4} + 20\,x^{3} + 174\,x^{2} + 980\,x + 2425\right)y^{6} + 39450\,x^{6} - 960\,x^{5} - 48\,\left(3\,x^{6} + 20\,x^{5} - 3\,x^{4} + 40\,x^{3} + 925\,x^{2} + 5940\,x + 14675\right)y^{4} - 39450\,x^{4} - 6000\,x^{3} + 96\,\left(x^{8} - x^{6} + 20\,x^{5} - 20\,x^{3} + 375\,x^{2} + 3000\,x + 7425\right)y^{2} - 9675\,x^{2} - 97000\,x - 240625}$$

In [15]:
```
N = Sig.scalar_field(sqrt(N2.simplify_full()), name='N')
print N
N.display()
```

```
Scalar field N on the 3-dimensional differentiable manifold Sigma
```

Out[15]:

$$\begin{array}{llcl} N: & \Sigma & \longrightarrow & \mathbb{R} \\ & (x, y, \phi) & \longmapsto & \sqrt{\frac{x^{10}+20\,x^{9}+576\,\left(x^{2}-1\right)y^{8}+99\,x^{8}-40\,x^{7}+96\,\left(x^{4}+10\,x^{3}+24\,x^{2}-10\,x-25\right)y^{6}-350\cdots\left(3\,x^{6}+10\,x^{5}-3\,x^{4}+20\,x^{3}+125\,x^{2}-30\,x-125\right)y^{4}+350\,x^{4}+1000\,x^{3}+96\,\left(x^{8}-x^{6}+1\cdots+525\,x^{2}-500\,x-625}{x^{10}+40\,x^{9}+576\,\left(x^{2}-1\right)y^{8}+699\,x^{8}+7920\,x^{7}+96\,\left(x^{4}+20\,x^{3}+174\,x^{2}+980\,x+2425\right)\left(3\,x^{6}+20\,x^{5}-3\,x^{4}+40\,x^{3}+925\,x^{2}+5940\,x+14675\right)y^{4}-39450\,x^{4}-6000\,x^{3}+96\left(x^{8}-x^{6}+20\,x^{5}-20\,x^{3}+375\,x^{2}+3000\,x+7425\right)y^{2}-9675\,x^{2}-97000\,x-240625}} \end{array}$$

The coordinate expression of the scalar field $N$:

In [16]:
```
N.expr()
```

Out[16]:

$$\sqrt{\frac{x^{10} + 20\,x^{9} + 576\,\left(x^{2} - 1\right)y^{8} + 99\,x^{8} - 40\,x^{7} + 96\,\left(x^{4} + 10\,x^{3} + 24\,x^{2} - 10\,x - 25\right)y^{6} - 350\,x^{6} - 480\,x^{5} - 48\,\left(3\,x^{6} + 10\,x^{5} - 3\,x^{4} + 20\,x^{3} + 125\,x^{2} - 30\,x - 125\right)y^{4} + 350\,x^{4} + 1000\,x^{3} + 96\,\left(x^{8} - x^{6} + 10\,x^{5} - 10\,x^{3} + 25\,x^{2} - 25\right)y^{2} + 525\,x^{2} - 500\,x - 625}{x^{10} + 40\,x^{9} + 576\,\left(x^{2} - 1\right)y^{8} + 699\,x^{8} + 7920\,x^{7} + 96\,\left(x^{4} + 20\,x^{3} + 174\,x^{2} + 980\,x + 2425\right)y^{6} + 39450\,x^{6} - 960\,x^{5} - 48\,\left(3\,x^{6} + 20\,x^{5} - 3\,x^{4} + 40\,x^{3} + 925\,x^{2} + 5940\,x + 14675\right)y^{4} - 39450\,x^{4} - 6000\,x^{3} + 96\,\left(x^{8} - x^{6} + 20\,x^{5} - 20\,x^{3} + 375\,x^{2} + 3000\,x + 7425\right)y^{2} - 9675\,x^{2} - 97000\,x - 240625}}$$

In [17]:
```
b3 = 2*m*q*CC2*(y^2-1)/(BB2*(m^2*(y^2-1)*(p^2*BB2^2*(x^2-1)
        +4*q^2*d^2*CC2^2*(y^2-1))/(AA2*BB2*d^2)))
b = Sig.vector_field('beta', latex_name=r'\beta')
b[3] = b3.simplify_full()
# unset components are zero
b.display_comp(only_nonzero=False)
```

Out[17]:

$$\beta^x = 0$$

$$\beta^y = 0$$

$$\beta^\phi = -\frac{\begin{array}{c}400\,\left(2\,\sqrt{6}x^7+24\,\left(\sqrt{6}x+5\,\sqrt{6}\right)y^6+20\,\sqrt{6}x^6-\sqrt{6}x^5-72\,\left(\sqrt{6}x+5\,\sqrt{6}\right)y^4-25\,\sqrt{6}x^4\right. \\ \left.-\left(\sqrt{6}x^5+15\,\sqrt{6}x^4+2\,\sqrt{6}x^3-10\,\sqrt{6}x^2-75\,\sqrt{6}x-365\,\sqrt{6}\right)y^2+10\,\sqrt{6}x^2-25\,\sqrt{6}x-125\,\sqrt{6}\right)\end{array}}{x^{10}+40\,x^9+576\,\left(x^2-1\right)y^8+699\,x^8+7920\,x^7+96\,\left(x^4+20\,x^3+174\,x^2+980\,x+2425\right)y^6+39450\,x^6-960}$$

$$\left(3\,x^6+20\,x^5-3\,x^4+40\,x^3+925\,x^2+5940\,x+14675\right)y^4-39450\,x^4-6000\,x^3+96$$

$$\left(x^8-x^6+20\,x^5-20\,x^3+375\,x^2+3000\,x+7425\right)y^2-9675\,x^2-97000\,x-240625$$

### Extrinsic curvature of $\Sigma$

We use the formula

$$K_{ij} = \frac{1}{2N}\mathcal{L}_\beta \gamma_{ij},$$

which is valid for any stationary spacetime:

In [18]:
```
K = gam.lie_derivative(b) / (2*N)
K.set_name('K')
print K
```

```
Field of symmetric bilinear forms K on the 3-dimensional differentiable
 manifold Sigma
```

The component $K_{13} = K_{x\phi}$:

In [19]: 

```
K[1,3]
```

Out[19]:

$$2$$

$$\left(6\sqrt{3}\sqrt{2}x^{16} - 13824\left(\sqrt{3}\sqrt{2}x^{2} + 10\sqrt{3}\sqrt{2}x + \sqrt{3}\sqrt{2}\right)y^{16} + 240\sqrt{3}\sqrt{2}x^{15}\right.$$

$$+ 3793\sqrt{3}\sqrt{2}x^{14} - 6912$$

$$\left(\sqrt{3}\sqrt{2}x^{4} + 20\sqrt{3}\sqrt{2}x^{3} + 150\sqrt{3}\sqrt{2}x^{2} + 500\sqrt{3}\sqrt{2}x + 817\sqrt{3}\sqrt{2}\right)y^{14}$$

$$+ 27650\sqrt{3}\sqrt{2}x^{13} + 72403\sqrt{3}\sqrt{2}x^{12} + 576$$

$$\left(27\sqrt{3}\sqrt{2}x^{6} + 310\sqrt{3}\sqrt{2}x^{5} + 1033\sqrt{3}\sqrt{2}x^{4} + 1060\sqrt{3}\sqrt{2}x^{3} + 10493\sqrt{3}\sqrt{2}x^{2}\right.$$

$$\left.+ 44870\sqrt{3}\sqrt{2}x + 69503\sqrt{3}\sqrt{2}\right)$$

$$- 81820\sqrt{3}\sqrt{2}x^{11} - 374975\sqrt{3}\sqrt{2}x^{10} - 96$$

$$\left(109\sqrt{3}\sqrt{2}x^{8} + 520\sqrt{3}\sqrt{2}x^{7} + 1504\sqrt{3}\sqrt{2}x^{6} + 19360\sqrt{3}\sqrt{2}x^{5} + 92770\right. \quad y^{1}$$

$$\sqrt{3}\sqrt{2}x^{4} + 157960\sqrt{3}\sqrt{2}x^{3} + 148264\sqrt{3}\sqrt{2}x^{2} + 731920\sqrt{3}\sqrt{2}x + 1256425$$

$$\left.\sqrt{3}\sqrt{2}\right)$$

$$- 313810\sqrt{3}\sqrt{2}x^{9} + 669975\sqrt{3}\sqrt{2}x^{8} + 24$$

$$\left(9\sqrt{3}\sqrt{2}x^{10} + 250\sqrt{3}\sqrt{2}x^{9} + 6873\sqrt{3}\sqrt{2}x^{8} + 40920\sqrt{3}\sqrt{2}x^{7} + 63402\sqrt{3}\sqrt{2}x\right.$$

$$+ 146220\sqrt{3}\sqrt{2}x^{5} + 1047426\sqrt{3}\sqrt{2}x^{4} + 2249400\sqrt{3}\sqrt{2}x^{3} + 876525\sqrt{3}\sqrt{2}x^{2}$$

$$\left.+ 4308810\sqrt{3}\sqrt{2}x + 8401925\sqrt{3}\sqrt{2}\right)$$

$$+ 1617000\sqrt{3}\sqrt{2}x^{7} + 999675\sqrt{3}\sqrt{2}x^{6} + 96$$

$$\left(20\sqrt{3}\sqrt{2}x^{11} - 179\sqrt{3}\sqrt{2}x^{10} - 50\sqrt{3}\sqrt{2}x^{9} - 2897\sqrt{3}\sqrt{2}x^{8} - 28400\sqrt{3}\sqrt{2}x^{7}\right.$$

$$- 57446\sqrt{3}\sqrt{2}x^{6} - 9020\sqrt{3}\sqrt{2}x^{5} - 237650\sqrt{3}\sqrt{2}x^{4} - 731060\sqrt{3}\sqrt{2}x^{3}$$

$$\left.- 267175\sqrt{3}\sqrt{2}x^{2} - 1037250\sqrt{3}\sqrt{2}x - 2111325\sqrt{3}\sqrt{2}\right)$$

$$- 2277250\sqrt{3}\sqrt{2}x^{5} - 4979375\sqrt{3}\sqrt{2}x^{4}$$

$$- \left(187\sqrt{3}\sqrt{2}x^{14} + 3590\sqrt{3}\sqrt{2}x^{13} - 5207\sqrt{3}\sqrt{2}x^{12} - 73540\sqrt{3}\sqrt{2}x^{11} - 4546\right.$$

$$\sqrt{3}\sqrt{2}x^{10} - 1150150\sqrt{3}\sqrt{2}x^{9} + 199401\sqrt{3}\sqrt{2}x^{8} - 1059000\sqrt{3}\sqrt{2}x^{7}$$

$$- 7811175\sqrt{3}\sqrt{2}x^{6} + 2899610\sqrt{3}\sqrt{2}x^{5} + 1675075\sqrt{3}\sqrt{2}x^{4} - 32834500$$

$$\left.\sqrt{3}\sqrt{2}x^{3} - 24681575\sqrt{3}\sqrt{2}x^{2} - 69684250\sqrt{3}\sqrt{2}x - 122823125\sqrt{3}\sqrt{2}\right)$$

$$- 4037500\sqrt{3}\sqrt{2}x^{3} + 3461875\sqrt{3}\sqrt{2}x^{2} - 6$$

$$\left(\sqrt{3}\sqrt{2}x^{16} + 40\sqrt{3}\sqrt{2}x^{15} + 601\sqrt{3}\sqrt{2}x^{14} + 4010\sqrt{3}\sqrt{2}x^{13} + 12935\sqrt{3}\sqrt{2}x^{12}\right.$$

$$- 1060\sqrt{3}\sqrt{2}x^{11} + 10449\sqrt{3}\sqrt{2}x^{10} + 139590\sqrt{3}\sqrt{2}x^{9} + 57825\sqrt{3}\sqrt{2}x^{8}$$

$$+ 146960\sqrt{3}\sqrt{2}x^{7} + 781475\sqrt{3}\sqrt{2}x^{6} - 702250\sqrt{3}\sqrt{2}x^{5} - 2108075\sqrt{3}\sqrt{2}x^{4}$$

$$\left.- 348500\sqrt{3}\sqrt{2}x^{3} + 2381875\sqrt{3}\sqrt{2}x^{2} + 5456250\sqrt{3}\sqrt{2}x + 6941250\sqrt{3}\sqrt{2}\right)$$

$$\left.+ 7231250\sqrt{3}\sqrt{2}x + 6109375\sqrt{3}\sqrt{2}\right)$$

The type-(1,1) tensor $K^\sharp$ of components $K^i_{\ j} = \gamma^{ik} K_{kj}$:

In [20]:
```
Ku = K.up(gam, 0)
print Ku
```

```
Tensor field of type (1,1) on the 3-dimensional differentiable manifold
 Sigma
```

We may check that the hypersurface $\Sigma$ is maximal, i.e. that $K^k_{\ k} = 0$:

In [21]:
```
trK = Ku.trace()
trK
```

Out[21]: $0$

### Connection and curvature

Let us call $D$ the Levi-Civita connection associated with $\gamma$:

In [22]:
```
D = gam.connection(name='D')
print D
```

```
Levi-Civita connection D associated with the Riemannian metric gam on t
he 3-dimensional differentiable manifold Sigma
```

The Ricci tensor associated with $\gamma$:

In [23]:
```
Ric = gam.ricci()
print Ric
```

```
Field of symmetric bilinear forms Ric(gam) on the 3-dimensional differe
ntiable manifold Sigma
```

The scalar curvature $R = \gamma^{ij} R_{ij}$:

In [24]:
```
R = gam.ricci_scalar(name='R')
print R
```

```
Scalar field R on the 3-dimensional differentiable manifold Sigma
```

### Terms related to the extrinsic curvature

Let us first evaluate the term $K_{ij} K^{ij}$:

In [25]:
```
Kuu = Ku.up(gam, 1)
trKK = K['_ij']*Kuu['^ij']
print trKK
```

```
Scalar field on the 3-dimensional differentiable manifold Sigma
```

Then we compute the symmetric bilinear form $k_{ij} := K_{ik} K^k_{\ j}$:

In [26]:
```
KK = K['_ik']*Ku['^k_j']
print KK
```

```
Tensor field of type (0,2) on the 3-dimensional differentiable manifold
 Sigma
```

We check that this tensor field is symmetric:

In [27]:
```
KK1 = KK.symmetrize()
KK == KK1
```

Out[27]: True

Accordingly, we work with the explicitly symmetric version:

In [28]:
```
KK = KK1
print KK
```

```
Field of symmetric bilinear forms on the 3-dimensional differentiable m
anifold Sigma
```

## Electric and magnetic parts of the Weyl tensor

The electric part is the bilinear form $E$ given by

$$E_{ij} = R_{ij} + KK_{ij} - K_{ik}K^k_{\ \ j}$$

In [29]:
```
E = Ric + trK*K - KK
print E
```

```
Field of symmetric bilinear forms on the 3-dimensional differentiable m
anifold Sigma
```

The magnetic part is the bilinear form $B$ defined by

$$B_{ij} = \epsilon^k_{\ \ li} D_k K^l_{\ \ j},$$

where $\epsilon^k_{\ \ li}$ are the components of the type-(1,2) tensor $\epsilon^\sharp$, related to the Levi-Civita alternating tensor $\epsilon$ associated with $\gamma$ by $\epsilon^k_{\ \ li} = \gamma^{km}\epsilon_{mli}$. In SageManifolds, $\epsilon$ is obtained by the command `volume_form()` and $\epsilon^\sharp$ by the command `volume_form(1)` (1 = 1 index raised):

In [30]:
```
eps = gam.volume_form()
print eps
```

```
3-form eps_gam on the 3-dimensional differentiable manifold Sigma
```

In [31]:
```
epsu = gam.volume_form(1)
print epsu
```

```
Tensor field of type (1,2) on the 3-dimensional differentiable manifold
 Sigma
```

In [32]:
```
DKu = D(Ku)
B = epsu['^k_li']*DKu['^l_jk']
print B
```

```
Tensor field of type (0,2) on the 3-dimensional differentiable manifold
 Sigma
```

Let us check that $B$ is symmetric:

In [33]:
```
B1 = B.symmetrize()
B == B1
```

Out[33]: True

Accordingly, we set

In [34]:
```
B = B1
B.set_name('B')
print B
```

```
Field of symmetric bilinear forms B on the 3-dimensional differentiable
 manifold Sigma
```

## 3+1 decomposition of the Simon-Mars tensor

We proceed according to the computation presented in arXiv:1412.6542.

Tensor $E^\sharp$ of components $E^i{}_j$:

In [35]:
```
Eu = E.up(gam, 0)
print Eu
```

```
Tensor field of type (1,1) on the 3-dimensional differentiable manifold
 Sigma
```

Tensor $B^\sharp$ of components $B^i{}_j$:

In [36]:
```
Bu = B.up(gam, 0)
print Bu
```

```
Tensor field of type (1,1) on the 3-dimensional differentiable manifold
 Sigma
```

1-form $\beta^\flat$ of components $\beta_i$ and its exterior derivative:

In [37]:
```
bd = b.down(gam)
xdb = bd.exterior_derivative()
print xdb
```

```
2-form on the 3-dimensional differentiable manifold Sigma
```

Scalar square of shift $\beta_i \beta^i$:

In [38]:
```
b2 = bd(b)
print b2
```

```
Scalar field on the 3-dimensional differentiable manifold Sigma
```

Scalar $Y = E(\beta, \beta) = E_{ij}\beta^i\beta^j$:

In [39]:
```
Ebb = E(b,b)
Y = Ebb
print Y
```

```
Scalar field on the 3-dimensional differentiable manifold Sigma
```

Scalar $\bar{Y} = B(\beta, \beta) = B_{ij}\beta^i\beta^j$:

In [40]:
```
Bbb = B(b,b)
Y_bar = Bbb
print Y_bar
```

```
Scalar field B(beta,beta) on the 3-dimensional differentiable manifold
Sigma
```

1-form of components $Eb_i = E_{ij}\beta^j$:

In [41]:
```
Eb = E.contract(b)
print Eb
```

```
1-form on the 3-dimensional differentiable manifold Sigma
```

Vector field of components $Eub^i = E^i{}_j\beta^j$:

In [42]:
```
Eub = Eu.contract(b)
print Eub
```

```
Vector field on the 3-dimensional differentiable manifold Sigma
```

1-form of components $Bb_i = B_{ij}\beta^j$:

In [43]:
```
Bb = B.contract(b)
print Bb
```

```
1-form on the 3-dimensional differentiable manifold Sigma
```

Vector field of components $Bub^i = B^i{}_j\beta^j$:

In [44]:
```
Bub = Bu.contract(b)
print Bub
```

```
Vector field on the 3-dimensional differentiable manifold Sigma
```

Vector field of components $Kub^i = K^i{}_j\beta^j$:

In [45]:
```
Kub = Ku.contract(b)
print Kub
```

```
Vector field on the 3-dimensional differentiable manifold Sigma
```

In [46]:
```
T = 2*b(N) - 2*K(b,b)
print T ; T.display()
```

```
Scalar field zero on the 3-dimensional differentiable manifold Sigma
```

Out[46]:

$$\begin{array}{llcl} 0: & \Sigma & \longrightarrow & \mathbb{R} \\ & (x, y, \phi) & \longmapsto & 0 \end{array}$$

In [47]:
```
Db = D(b)  #  Db^i_j = D_j b^i
Dbu = Db.up(gam, 1)  # Dbu^{ij} = D^j b^i
bDb = b*Dbu  # bDb^{ijk} = b^i D^k b^j
T_bar = eps['_ijk']*bDb['^ikj']
print T_bar ; T_bar.display()
```

```
Scalar field zero on the 3-dimensional differentiable manifold Sigma
```

Out[47]:
$$\begin{array}{llll} 0: & \Sigma & \longrightarrow & \mathbb{R} \\ & (x,y,\phi) & \longmapsto & 0 \end{array}$$

In [48]:
```
epsb = eps.contract(b)
print epsb
```

```
2-form on the 3-dimensional differentiable manifold Sigma
```

In [49]:
```
epsB = eps['_ijl']*Bu['^l_k']
print epsB
```

```
Tensor field of type (0,3) on the 3-dimensional differentiable manifold
 Sigma
```

In [50]:
```
Z = 2*N*( D(N)  -K.contract(b)) + b.contract(xdb)
print Z
```

```
1-form on the 3-dimensional differentiable manifold Sigma
```

In [51]:
```
DNu = D(N).up(gam)
A = 2*(DNu - Ku.contract(b))*b + N*Dbu
Z_bar = eps['_ijk']*A['^kj']
print Z_bar
```

```
1-form on the 3-dimensional differentiable manifold Sigma
```

In [52]:
```
W = N*Eb + epsb.contract(Bub)
print W
```

```
1-form on the 3-dimensional differentiable manifold Sigma
```

In [53]:
```
W_bar = N*Bb - epsb.contract(Eub)
print W_bar
```

```
1-form on the 3-dimensional differentiable manifold Sigma
```

In [54]:
```
M = - 4*Eb(Kub - DNu) - 2*(epsB['_ij.']*Dbu['^ji'])(b)
print M ; M.display()
```

```
Scalar field zero on the 3-dimensional differentiable manifold Sigma
```

Out[54]:
$$\begin{array}{llll} 0: & \Sigma & \longrightarrow & \mathbb{R} \\ & (x,y,\phi) & \longmapsto & 0 \end{array}$$

In [55]:
```
M_bar = 2*(eps.contract(Eub))['_ij']*Dbu['^ji'] - 4*Bb(Kub - DNu)
print M_bar ; M_bar.display()
```

```
Scalar field zero on the 3-dimensional differentiable manifold Sigma
```

Out[55]:
$$\begin{array}{llll} 0: & \Sigma & \longrightarrow & \mathbb{R} \\ & (x,y,\phi) & \longmapsto & 0 \end{array}$$

In [56]:
```
F = (N^2 - b2)*gam + bd*bd
print F
```

```
Field of symmetric bilinear forms on the 3-dimensional differentiable m
anifold Sigma
```

In [57]:
```
A = epsB['_ilk']*b['^l'] + epsB['_ikl']*b['^l'] \
    + Bu['^m_i']*epsb['_mk'] - 2*N*E
xdbE = xdb['_kl']*Eu['^k_i']
L = 2*N*epsB['_kli']*Dbu['^kl'] + 2*xdb['_ij']*Eub['^j'] \
    + 2*xdbE['_li']*b['^l'] + 2*A['_ik']*(Kub - DNu)['^k']
print L
```

```
1-form on the 3-dimensional differentiable manifold Sigma
```

In [58]:
```
N2pbb = N^2 + b2
V = N2pbb*E - 2*(b.contract(E)*bd).symmetrize() + Ebb*gam \
    + 2*N*(b.contract(epsB).symmetrize())
print V
```

```
Field of symmetric bilinear forms on the 3-dimensional differentiable m
anifold Sigma
```

In [59]:
```
beps = b.contract(eps)
V_bar = N2pbb*B - 2*(b.contract(B)*bd).symmetrize() + Bbb*gam \
        -2*N*(beps['_il']*Eu['^l_j']).symmetrize()
print V_bar
```

```
Field of symmetric bilinear forms on the 3-dimensional differentiable m
anifold Sigma
```

In [60]:
```
F = (N^2 - b2)*gam + bd*bd
print F
```

```
Field of symmetric bilinear forms on the 3-dimensional differentiable m
anifold Sigma
```

In [61]:
```
R1 = (4*(V*Z - V_bar*Z_bar) + F*L).antisymmetrize(1,2)
print R1
```

```
Tensor field of type (0,3) on the 3-dimensional differentiable manifold
 Sigma
```

In [62]:
```
R2 = 4*(T*V - T_bar*V_bar - W*Z + W_bar*Z_bar) + M*F - N*bd*L
print R2
```

```
Tensor field of type (0,2) on the 3-dimensional differentiable manifold
 Sigma
```

In [63]:
```
R3 = (4*(W*Z - W_bar*Z_bar) + N*bd*L).antisymmetrize()
print R3
```

```
2-form on the 3-dimensional differentiable manifold Sigma
```

In [64]:
```
R2[3,1] == -2*R3[3,1]
```

Out[64]: True

In [65]:
```
R2[3,2] == -2*R3[3,2]
```

Out[65]: True

In [66]:
```
R4 = 4*(T*W - T_bar*W_bar) -4*(Y*Z - Y_bar*Z_bar) + N*M*bd - b2*L
print R4
```

```
1-form on the 3-dimensional differentiable manifold Sigma
```

In [67]:
```
epsE = eps['_ijl']*Eu['^l_k']
print epsE
```

```
Tensor field of type (0,3) on the 3-dimensional differentiable manifold
 Sigma
```

In [68]:
```
A = - epsE['_ilk']*b['^l'] - epsE['_ikl']*b['^l'] \
    - Eu['^m_i']*epsb['_mk'] - 2*N*B
xdbB = xdb['_kl']*Bu['^k_i']
L_bar = - 2*N*epsE['_kli']*Dbu['^kl'] + 2*xdb['_ij']*Bub['^j'] \
        + 2*xdbB['_li']*b['^l'] + 2*A['_ik']*(Kub - DNu)['^k']
print L_bar
```

```
1-form on the 3-dimensional differentiable manifold Sigma
```

In [69]:
```
R1_bar = (4*(V*Z_bar + V_bar*Z) + F*L_bar).antisymmetrize(1,2)
print R1_bar
```

```
Tensor field of type (0,3) on the 3-dimensional differentiable manifold
 Sigma
```

In [70]:
```
R2_bar = 4*(T_bar*V + T*V_bar) - 4*(W*Z_bar + W_bar*Z) \
         + M_bar*F - N*bd*L_bar
print R2_bar
```

```
Tensor field of type (0,2) on the 3-dimensional differentiable manifold
 Sigma
```

In [71]:
```
R3_bar = (4*(W*Z_bar + W_bar*Z) + N*bd*L_bar).antisymmetrize()
print R3_bar
```

```
2-form on the 3-dimensional differentiable manifold Sigma
```

In [72]:
```
R4_bar = 4*(T_bar*W + T*W_bar - Y*Z_bar - Y_bar*Z) \
         + M_bar*N*bd - b2*L_bar
print R4_bar
```

```
1-form on the 3-dimensional differentiable manifold Sigma
```

In [73]:
```
R1u = R1.up(gam)
print R1u
```

```
Tensor field of type (3,0) on the 3-dimensional differentiable manifold
 Sigma
```

In [74]:
```
R2u = R2.up(gam)
print R2u
```

```
Tensor field of type (2,0) on the 3-dimensional differentiable manifold
 Sigma
```

In [75]:
```
R3u = R3.up(gam)
print R3u
```

```
Tensor field of type (2,0) on the 3-dimensional differentiable manifold
 Sigma
```

In [76]:
```
R4u = R4.up(gam)
print R4u
```

```
Vector field on the 3-dimensional differentiable manifold Sigma
```

In [77]:
```
R1_baru = R1_bar.up(gam)
print R1_baru
```

```
Tensor field of type (3,0) on the 3-dimensional differentiable manifold
 Sigma
```

In [78]:
```
R2_baru = R2_bar.up(gam)
print R2_baru
```

```
Tensor field of type (2,0) on the 3-dimensional differentiable manifold
 Sigma
```

In [79]:
```
R3_baru = R3_bar.up(gam)
print R3_baru
```

```
Tensor field of type (2,0) on the 3-dimensional differentiable manifold
 Sigma
```

In [80]:
```
R4_baru = R4_bar.up(gam)
print R4_baru
```

```
Vector field on the 3-dimensional differentiable manifold Sigma
```

### Simon-Mars scalars

In [81]:
```
S1 = 4*(R1['_ijk']*R1u['^ijk'] - R1_bar['_ijk']*R1_baru['^ijk'] \
     - 2*(R2['_ij']*R2u['^ij'] - R2_bar['_ij']*R2_baru['^ij']) \
     - R3['_ij']*R3u['^ij'] + R3_bar['_ij']*R3_baru['^ij'] \
     + 2*(R4['_i']*R4u['^i'] - R4_bar['_i']*R4_baru['^i']))
print S1
```

```
Scalar field on the 3-dimensional differentiable manifold Sigma
```

In [82]:
```
S1E = S1.expr()
```

In [83]:
```
S2 = 4*(R1['_ijk']*R1_baru['^ijk'] + R1_bar['_ijk']*R1u['^ijk'] \
     - 2*(R2['_ij']*R2_baru['^ij'] + R2_bar['_ij']*R2u['^ij']) \
     - R3['_ij']*R3_baru['^ij'] - R3_bar['_ij']*R3u['^ij'] \
     + 2*(R4['_i']*R4_baru['^i'] + R4_bar['_i']*R4u['^i']))
print S2
```

```
Scalar field on the 3-dimensional differentiable manifold Sigma
```

In [84]:
```
S2E = S2.expr()
```

In [85]:
```
lS1E = log(S1E,10).simplify_full()
```

In [86]:
```
lS2E = log(S2E,10).simplify_full()
```

Simon-Mars scalars expressed in terms of the coordinates $X = -1/x, y$:

In [87]:
```
var('X')
S1EX = S1E.subs(x=-1/X).simplify_full()
S2EX = S2E.subs(x=-1/X).simplify_full()
```

Definition of the ergoregion:

In [88]:
```
g00 = - AA2/BB2
g00X = g00.subs(x=-1/X).simplify_full()
```

In [89]:
```
ergXy = implicit_plot(g00X, (X,-1,0), (y,-1,1), plot_points=200,
                      fill=False, linewidth=1, color='black',
                      axes_labels=(r"$X\,\left[M^{-1}\right]$",
                                   r"$y\,\left[M\right]$"),
                      fontsize=14)
```

Due to the very high degree of the polynomials involved in the expression of the Simon-Mars scalars, the floating-point precision of Sage's `contour_plot` function (53 bits) is not sufficient. Taking avantage that Sage is **open-source**, we modify the function to allow for an arbitrary precision. First, we define a sampling function with a floating-point precision specified by the user (argument `precis`):

In [90]:
```
def array_precisXy(fXy, Xmin, Xmax, ymin, ymax, np, precis, tronc):
    RP = RealField(precis)
    Xmin = RP(Xmin)
    Xmax = RP(Xmax)
    ymin = RP(ymin)
    ymax = RP(ymax)
    dX = (Xmax - Xmin) / RP(np-1)
    dy = (ymax - ymin) / RP(np-1)
    resu = []
    for i in range(np):
        list_y = []
        yy = ymin + dy * RP(i)
        fyy = fXy.subs(y=yy)
        for j in range(np):
            XX = Xmin + dX * RP(j)
            fyyXX = fyy.subs(X = XX)
            val = RP(log(abs(fyyXX) + 1e-20, 10))
            if val < -tronc:
                val = -tronc
            elif val > tronc:
                val = tronc
            list_y.append(val)
        resu.append(list_y)
    return resu
```

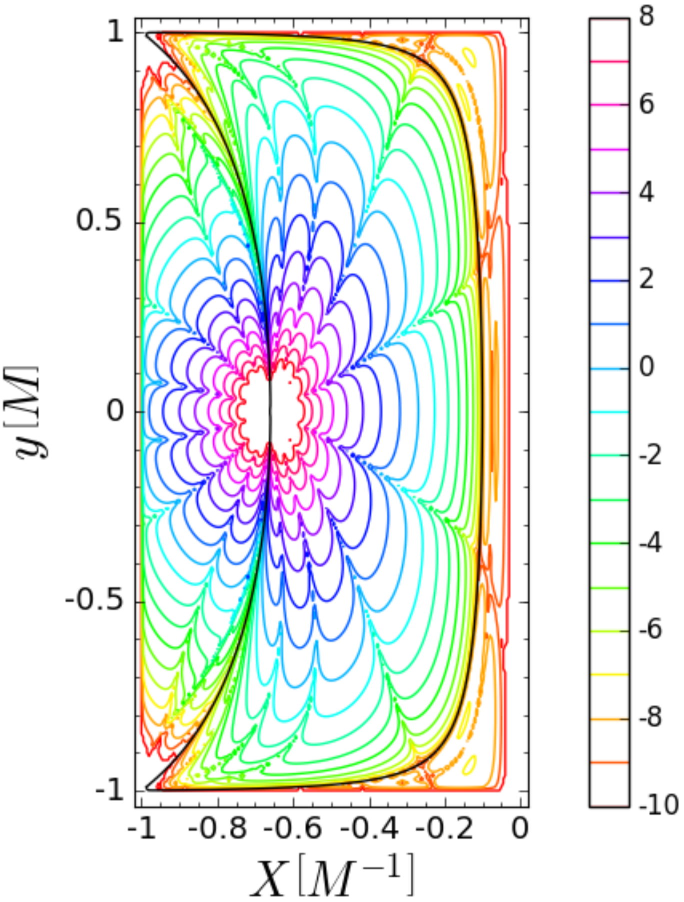
Then we redefine `contour_plot` so that it uses the sampling function with a floating-point precision of 200 bits:

In [91]:
```
from sage.misc.decorators import options, suboptions

@suboptions('colorbar', orientation='vertical', format=None,
            spacing=None)
@suboptions('label', fontsize=9, colors='blue', inline=None,
            inline_spacing=3, fmt="%1.2f")
@options(plot_points=100, fill=True, contours=None, linewidths=None,
         linestyles=None, labels=False, frame=True, axes=False,
         colorbar=False, legend_label=None, aspect_ratio=1)
def contour_plot_precisXy(f, xrange, yrange, **options):
    from sage.plot.all import Graphics
    from sage.plot.misc import setup_for_eval_on_grid
    from sage.plot.contour_plot import ContourPlot

    np = options['plot_points']
    precis = 200  # floating-point precision = 200 bits
    tronc = 10
    xy_data_array = array_precisXy(f, xrange[0], xrange[1],
                                   yrange[0], yrange[1], np, precis,
                                   tronc)

    g = Graphics()

    # Reset aspect_ratio to 'automatic' in case scale is 'semilog[xy]'.
    # Otherwise matplotlib complains.
    scale = options.get('scale', None)
    if isinstance(scale, (list, tuple)):
        scale = scale[0]
    if scale == 'semilogy' or scale == 'semilogx':
        options['aspect_ratio'] = 'automatic'

    g._set_extra_kwds(Graphics._extract_kwds_for_show(options,
                                    ignore=['xmin', 'xmax']))
    g.add_primitive(ContourPlot(xy_data_array, xrange,
                                yrange, options))
    return g
```

Then we are able to draw the contour plot of the two Simon-Mars scalars, in terms of the coordinates $(X, y)$ (Figure 11 of arXiv:1412.6542):

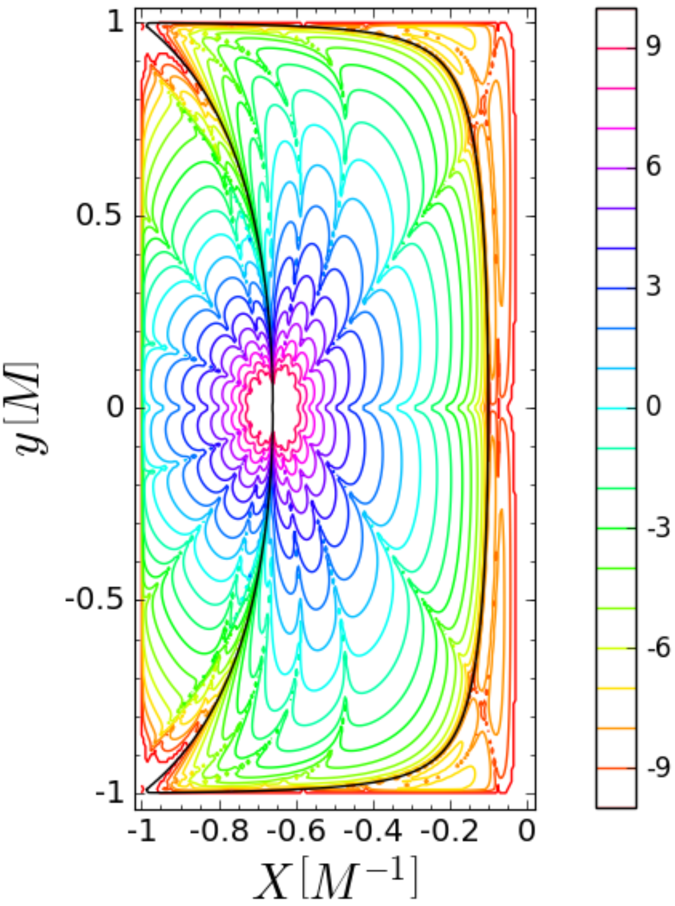
In [92]:
```
c1Xy = contour_plot_precisXy(S1EX, (-1,0), (-1,1),
                             plot_points=200,
                             fill=False, cmap='hsv',
                             linewidths=1,
                             contours=(-10,-9,-8,-7,-6,-5,-4,-3,
                                       -2,-1,0,1,2,3,4,5,6,7,8),
                             colorbar=True,
                             colorbar_spacing='uniform',
                             colorbar_format='%1.f',
                             axes_labels=(r"$X\,\left[M^{-1}\right]$",
                                          r"$y\,\left[M\right]$"),
                             fontsize=14)
```

In [93]:
```
S1TSXy = c1Xy+ergXy
show(S1TSXy)
```




In [94]:
```
c2Xy = contour_plot_precisXy(S2EX, (-1,0), (-1,1),
                             plot_points=200,
                             fill=False, cmap='hsv',
                             linewidths=1,
                             contours=(-10,-9,-8,-7,-6,-5,-4,-3,-2,
                                       -1,0,1,2,3,4,5,6,7,8,9,10),
                             colorbar=True,
                             colorbar_spacing='uniform',
                             colorbar_format='%1.f',
                             axes_labels=(r"$X\,\left[M^{-1}\right]$",
                                          r"$y\,\left[M\right]$"),
                             fontsize=14)
```

In [95]:
```
S2TSXy = c2Xy + ergXy
show(S2TSXy)
```




Let us do the same in terms of the Weyl-Lewis-Papapetrou cylindrical coordinates $(\rho, z)$, which are related to the prolate spheroidal coordinates $(x, y)$ by

$$\rho = \sqrt{(x^2 - 1)(1 - y^2)} \quad \text{and} \quad z = xy.$$

For simplicity, we denote $\rho$ by $r$:

In [96]:
```
var('r z')
```

Out[96]: $(r, z)$

In [97]:
```
S1Erz = S1E.subs(x=1/2*(sqrt(r^2+(z+1)^2)+sqrt(r^2+(z-1)^2)),
                 y=1/2*(sqrt(r^2+(z+1)^2)-sqrt(r^2+(z-1)^2)))
S1Erz = S1Erz.simplify_full()
```

In [98]:
```
S2Erz = S2E.subs(x=1/2*(sqrt(r^2+(z+1)^2)+sqrt(r^2+(z-1)^2)),
                 y=1/2*(sqrt(r^2+(z+1)^2)-sqrt(r^2+(z-1)^2)))
S2Erz = S2Erz.simplify_full()
```

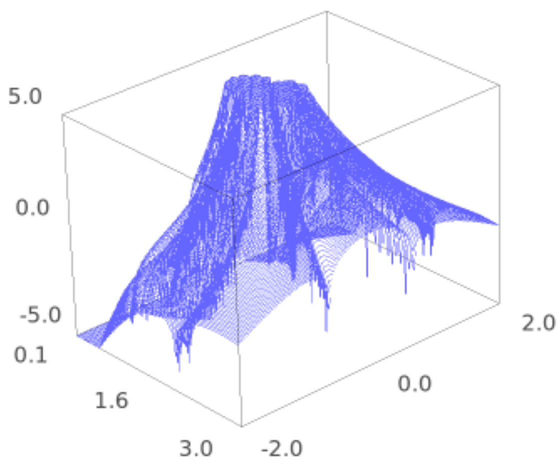
In [99]:
```
def tab_precis(fz, zz, rmin, rmax, np, precis, tronc):
    RP = RealField(precis)
    rmin = RP(rmin)
    rmax = RP(rmax)
    zz = RP(zz)
    dr = (rmax - rmin) / RP(np-1)
    resu = []
    fzz = fz.subs(z=zz)
    for i in range(np):
        rr = rmin + dr * RP(i)
        val = RP(log(abs(fzz.subs(r = rr)), 10))
        if val < -tronc:
            val = -tronc
        elif val > tronc:
            val = tronc
        resu.append((rr, zz, val))
    return resu
```

### 3D plots

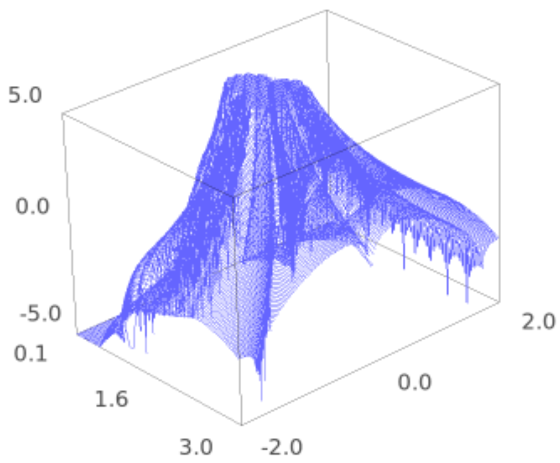
We also a viewer for 3D plots (use `'threejs'` or `'jmol'` for interactive 3D graphics):

In [109]:
```
viewer3D = 'jmol' # must be 'threejs', jmol', 'tachyon' or None (defaul
t)
```

In [110]:
```
gg = Graphics()
rmin = 0.1
rmax = 3
zmin = -2
zmax = 2
npr = 200
npz = npr
precis = 200 # 200-bits floating-point precision
tronc = 5
dz = (zmax-zmin) / (npz-1)
for i in range(npz):
    zz = zmin + i*dz
    gg += line3d(tab_precis(S1Erz, zz, rmin, rmax,
                            npr, precis, tronc))
show(gg, viewer=viewer3D, axes_labels=['rho', 'z', 'S_1'],
     aspect_ratio=[1,1,0.3])
```



In [111]:
```
gg2 = Graphics()
for i in range(npz):
    zz = zmin + i*dz
    gg2 += line3d(tab_precis(S2Erz, zz, rmin, rmax,
                             npr, precis, tronc))
show(gg2, viewer=viewer3D, axes_labels=['rho', 'z', 'S_2'],
     aspect_ratio=[1,1,0.3])
```




## 2D contour plots

Contour plots of the two Simon-Mars scalar fields in terms of coordinates $(\rho, z)$ (Figure 12 of [arXiv:1412.6542](arXiv:1412.6542))

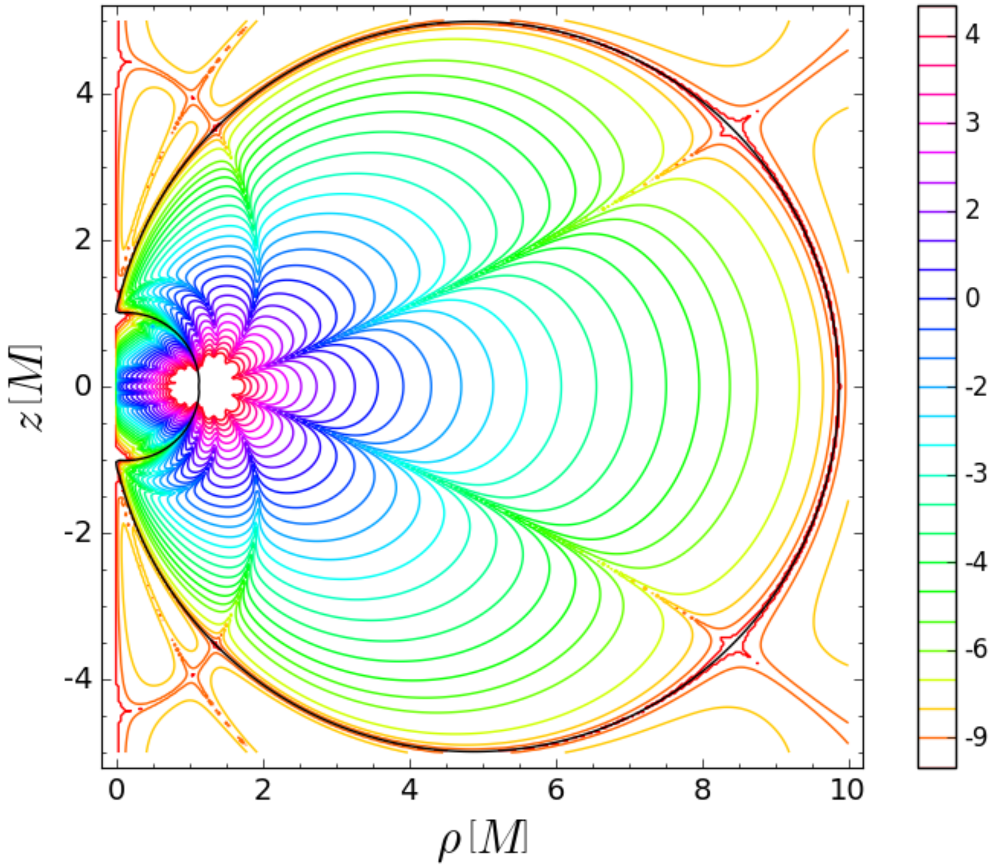
In [103]:
```
def array_precis(frz, rmin, rmax, zmin, zmax, np, precis,
                 tronc):
    RP = RealField(precis)
    rmin = RP(rmin)
    rmax = RP(rmax)
    zmin = RP(zmin)
    zmax = RP(zmax)
    dr = (rmax - rmin) / RP(np-1)
    dz = (zmax - zmin) / RP(np-1)
    resu = []
    for i in range(np):
        list_z = []
        zz = zmin + dz * RP(i)
        fzz = frz.subs(z=zz)
        for j in range(np):
            rr = rmin + dr * RP(j)
            fzzrr = fzz.subs(r = rr)
            val = RP(log(abs(fzzrr) + 1e-20, 10))
            if val < -tronc:
                val = -tronc
            elif val > tronc:
                val = tronc
            list_z.append(val)
        resu.append(list_z)
    return resu
```

In [104]:
```
rmin = 0.1
rmax = 3
zmin = -2
zmax = 2
npr = 10
npz = npr
precis = 100
tronc = 5
val = array_precis(S1Erz, rmin, rmax, zmin, zmax, npr,
                   precis, tronc)
```

In [105]:
```
from sage.misc.decorators import options, suboptions

@suboptions('colorbar', orientation='vertical', format=None,
            spacing=None)
@suboptions('label', fontsize=9, colors='blue', inline=None,
            inline_spacing=3, fmt="%1.2f")
@options(plot_points=100, fill=True, contours=None,
         linewidths=None, linestyles=None, labels=False,
         frame=True, axes=False, colorbar=False,
         legend_label=None, aspect_ratio=1)
def contour_plot_precis(f, xrange, yrange, **options):
    from sage.plot.all import Graphics
    from sage.plot.misc import setup_for_eval_on_grid
    from sage.plot.contour_plot import ContourPlot

    np = options['plot_points']
    precis = 200
    tronc = 10
    xy_data_array = array_precis(f, xrange[0], xrange[1],
                                 yrange[0], yrange[1], np,
                                 precis, tronc)

    g = Graphics()

    # Reset aspect_ratio to 'automatic' in case scale is 'semilog[xy]'.
    # Otherwise matplotlib complains.
    scale = options.get('scale', None)
    if isinstance(scale, (list, tuple)):
        scale = scale[0]
    if scale == 'semilogy' or scale == 'semilogx':
        options['aspect_ratio'] = 'automatic'

    g._set_extra_kwds(Graphics._extract_kwds_for_show(options,
                                     ignore=['xmin', 'xmax']))
    g.add_primitive(ContourPlot(xy_data_array, xrange, yrange,
                                options))
    return g
```

In [106]:
```
c1rz = contour_plot_precis(S1Erz, (0.0001,10), (-5,5.001),
                           plot_points=300, fill=False,
                           cmap='hsv', linewidths=1,
                           contours=(-10,-9,-8,-7,-6,-5.5,-5,-4.5,
                                     -4,-3.5,-3,-2.5,-2,-1.5,-1,
                                     -0.5,0,0.5,1,1.5,2,2.5,3,3.5,
                                     4,4.5,5),
                           colorbar=True,
                           colorbar_spacing='uniform',
                           colorbar_format='%1.f',
                           axes_labels=(r"$\rho\,\left[M\right]$",
                                        r"$z\,\left[M\right]$"),
                           fontsize=14)
```

In [107]:
```
g00rz = g00.subs(x=1/2*(sqrt(r^2+(z+1)^2)+sqrt(r^2+(z-1)^2)),
                 y=1/2*(sqrt(r^2+(z+1)^2)
                        -sqrt(r^2+(z-1)^2))).simplify_full()
c2 = implicit_plot(g00rz, (r,0.0001,10), (z,-5,5.001),
                   plot_points=200, fill=False,
                   linewidth=1, color='black',
                   axes_labels=(r"$\rho\,\left[M\right]$",
                                r"$z\,\left[M\right]$"),
                   fontsize=14)
S1TSrz = c1rz+c2
show(S1TSrz)
```



In [108]:
```
c2rz = contour_plot_precis(S2Erz, (0.0001,10), (-5,5.001),
                           plot_points=300, fill=False,
                           cmap='hsv', linewidths=1,
                           contours=(-10,-9,-8,-7,-6,-5.5,-5,-4.5,
                                     -4,-3.5,-3,-2.5,-2,-1.5,-1,
                                     -0.5,0,0.5,1,1.5,2,2.5,3,3.5,
                                     4,4.5,5),
                           colorbar=True,
                           colorbar_spacing='uniform',
                           colorbar_format='%1.f',
                           axes_labels=(r"$\rho\,\left[M\right]$",
                                        r"$z\,\left[M\right]$"),
                           fontsize=14)
S2TSrz = c2rz+c2
show(S2TSrz)
```