Simon-Mars tensor and Kerr spacetime

This worksheet demonstrates a few capabilities of SageManifolds (version 1.0, as included in SageMath 7.5) regarding the computation of the Simon-Mars tensor.

Click here to download the worksheet file (ipynb format). To run it, you must start SageMath within the Jupyter notebook, via the command `sage -n jupyter`

NB: a version of SageMath at least equal to 7.5 is required to run this worksheet:

In [1]:
```
version()
```

Out[1]:
```
'SageMath version 7.5.1, Release Date: 2017-01-15'
```

First we set up the notebook to display mathematical objects using LaTeX rendering:

In [2]:
```
%display latex
```

Since some computations are quite long, we ask for running them in parallel on 8 cores:

In [3]:
```
Parallelism().set(nproc=8)
```

Spacetime manifold

We declare the Kerr spacetime (or more precisely the part of the Kerr spacetime covered by Boyer-Lindquist coordinates) as a 4-dimensional manifold $\mathcal{M}$:

In [4]:
```
M = Manifold(4, 'M', latex_name=r'\mathcal{M}')
print(M)
```

```
4-dimensional differentiable manifold M
```

The standard **Boyer-Lindquist coordinates** (t, r, θ, ϕ) are introduced by declaring a chart X on $\mathcal{M}$, via the method `chart()`, the argument of which is a string expressing the coordinates names, their ranges (the default is $(-\infty, +\infty)$) and their LaTeX symbols:

In [5]:
```
X.<t,r,th,ph> = M.chart(r't r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi
')
print(X) ; X
```

```
Chart (M, (t, r, th, ph))
```

Out[5]: $(\mathcal{M}, (t, r, \theta, \phi))$

Metric tensor

The 2 parameters m and a of the Kerr spacetime are declared as symbolic variables:

In [6]:
```
var('m, a', domain='real')
```

Out[6]: (m, a)

Let us introduce the spacetime metric g and set its components in the coordinate frame associated with Boyer-Lindquist coordinates, which is the current manifold's default frame:

In [7]:
```
g = M.lorentzian_metric('g')
rho2 = r^2 + (a*cos(th))^2
Delta = r^2 -2*m*r + a^2
g[0,0] = -(1-2*m*r/rho2)
g[0,3] = -2*a*m*r*sin(th)^2/rho2
g[1,1], g[2,2] = rho2/Delta, rho2
g[3,3] = (r^2+a^2+2*m*r*(a*sin(th))^2/rho2)*sin(th)^2
g.display()
```

Out[7]:

$$g = \left(\frac{2\,mr}{a^2\cos(\theta)^2 + r^2} - 1\right) dt \otimes dt + \left(-\frac{2\,amr\sin(\theta)^2}{a^2\cos(\theta)^2 + r^2}\right) dt \otimes d\phi + \left(\frac{a^2\cos(\theta)^2 + r^2}{a^2 - 2\,mr + r^2}\right) dr \otimes dr + \left(a^2\cos(\theta)^2 + r^2\right) d\theta \otimes d\theta + \left(-\frac{2\,amr\sin(\theta)^2}{a^2\cos(\theta)^2 + r^2}\right) d\phi \otimes dt + \left(\frac{2\,a^2mr\sin(\theta)^2}{a^2\cos(\theta)^2 + r^2} + a^2 + r^2\right)\sin(\theta)^2 d\phi \otimes d\phi$$

In [8]:
```
g[:]
```

Out[8]:

$$\left(\begin{array}{cccc}
\frac{2\,mr}{a^2\cos(\theta)^2+r^2} - 1 & 0 & 0 & -\frac{2\,amr\sin(\theta)^2}{a^2\cos(\theta)^2+r^2} \\
0 & \frac{a^2\cos(\theta)^2+r^2}{a^2-2\,mr+r^2} & 0 & 0 \\
0 & 0 & a^2\cos(\theta)^2 + r^2 & 0 \\
-\frac{2\,amr\sin(\theta)^2}{a^2\cos(\theta)^2+r^2} & 0 & 0 & \left(\frac{2\,a^2mr\sin(\theta)^2}{a^2\cos(\theta)^2+r^2} + a^2 + r^2\right)\sin(\theta)^2
\end{array}\right)$$

The Levi-Civita connection ∇ associated with g:

In [9]:
```
nabla = g.connection() ; print(nabla)
```

```
Levi-Civita connection nabla_g associated with the Lorentzian metric g
on the 4-dimensional differentiable manifold M
```

As a check, we verify that the covariant derivative of g with respect to ∇ vanishes identically:

In [10]:
```
nabla(g).display()
```

Out[10]:

$$\nabla_g g = 0$$

Killing vector

The default vector frame on the spacetime manifold is the coordinate basis associated with Boyer-Lindquist coordinates:

In [11]:
```
M.default_frame() is X.frame()
```
Out[11]: True

In [12]:
```
X.frame()
```
Out[12]:
$$\left(\mathcal{M}, \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\right)\right)$$

Let us consider the first vector field of this frame:

In [13]:
```
xi = X.frame()[0] ; xi
```
Out[13]:
$$\frac{\partial}{\partial t}$$

In [14]:
```
print(xi)
```
```
Vector field d/dt on the 4-dimensional differentiable manifold M
```

The 1-form associated to it by metric duality is

In [15]:
```
xi_form = xi.down(g)
xi_form.set_name('xi_form', r'\underline{\xi}')
print(xi_form) ; xi_form.display()
```
```
1-form xi_form on the 4-dimensional differentiable manifold M
```
Out[15]:
$$\underline{\xi} = \left(-\frac{a^2 \cos(\theta)^2 - 2\,mr + r^2}{a^2 \cos(\theta)^2 + r^2}\right) \mathrm{d}t + \left(-\frac{2\,amr \sin(\theta)^2}{a^2 \cos(\theta)^2 + r^2}\right) \mathrm{d}\phi$$

Its covariant derivative is

In [16]:
```
nab_xi = nabla(xi_form)
print(nab_xi) ; nab_xi.display()
```

```
Tensor field nabla_g(xi_form) of type (0,2) on the 4-dimensional differ
entiable manifold M
```

Out[16]:

$$
\begin{aligned}
\nabla_g \underline{\xi} = & \left(\frac{a^2 m \cos(\theta)^2 - mr^2}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}t \otimes \mathrm{d}r \\
& + \left(\frac{2\, a^2 mr \cos(\theta) \sin(\theta)}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}t \otimes \mathrm{d}\theta \\
& + \left(-\frac{a^2 m \cos(\theta)^2 - mr^2}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}r \otimes \mathrm{d}t \\
& + \left(\frac{\left(a^3 m \cos(\theta)^2 - amr^2\right) \sin(\theta)^2}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}r \otimes \mathrm{d}\phi \\
& + \left(-\frac{2\, a^2 mr \cos(\theta) \sin(\theta)}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}\theta \otimes \mathrm{d}t \\
& + \left(\frac{2\left(a^3 mr + amr^3\right) \cos(\theta) \sin(\theta)}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}\theta \otimes \mathrm{d}\phi \\
& + \left(-\frac{\left(a^3 m \cos(\theta)^2 - amr^2\right) \sin(\theta)^2}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}\phi \otimes \mathrm{d}r \\
& + \left(-\frac{2\left(a^3 mr + amr^3\right) \cos(\theta) \sin(\theta)}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}\phi \otimes \mathrm{d}\theta
\end{aligned}
$$

Let us check that the Killing equation is satisfied:

In [17]:
```
nab_xi.symmetrize().display()
```

Out[17]: 0

Equivalently, we check that the Lie derivative of the metric along ξ vanishes:

In [18]:
```
g.lie_der(xi).display()
```

Out[18]: 0

Thank to Killing equation, $\nabla_g \underline{\xi}$ is antisymmetric. We may therefore define a 2-form by $F := -\nabla_g \xi$. Here we enforce the antisymmetry by calling the function `antisymmetrize()` on `nab_xi`:

In [19]:
```
F = - nab_xi.antisymmetrize()
F.set_name('F')
print(F)
F.display()
```

```
2-form F on the 4-dimensional differentiable manifold M
```

Out[19]:

$$F = \left(-\frac{a^2 m \cos(\theta)^2 - m r^2}{a^4 \cos(\theta)^4 + 2 a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}t \wedge \mathrm{d}r + \left(-\frac{2 a^2 m r \cos(\theta) \sin(\theta)}{a^4 \cos(\theta)^4 + 2 a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}t \wedge \mathrm{d}\theta + \left(-\frac{\left(a^3 m \cos(\theta)^2 - a m r^2\right) \sin(\theta)^2}{a^4 \cos(\theta)^4 + 2 a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}r \wedge \mathrm{d}\phi + \left(-\frac{2\left(a^3 m r + a m r^3\right) \cos(\theta) \sin(\theta)}{a^4 \cos(\theta)^4 + 2 a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}\theta \wedge \mathrm{d}\phi$$

We check that

In [20]:
```
F == - nab_xi
```

Out[20]: True

The squared norm of the Killing vector is:

In [21]:
```
lamb = - g(xi,xi)
lamb.set_name('lambda', r'\lambda')
print(lamb)
lamb.display()
```

```
Scalar field lambda on the 4-dimensional differentiable manifold M
```

Out[21]:

$$\begin{array}{llcl} \lambda: & \mathcal{M} & \longrightarrow & \mathbb{R} \\ & (t, r, \theta, \phi) & \longmapsto & \frac{a^2 \cos(\theta)^2 - 2 m r + r^2}{a^2 \cos(\theta)^2 + r^2} \end{array}$$

Instead of invoking $g(\xi, \xi)$, we could have evaluated λ by means of the 1-form $\underline{\xi}$ acting on the vector field ξ:

In [22]:
```
lamb == - xi_form(xi)
```

Out[22]: True

or, using index notation as $\lambda = -\xi_a \xi^a$:

In [23]:
```
lamb == - ( xi_form['_a']*xi['^a'] )
```

Out[23]: True

Curvature

The Riemann curvature tensor associated with g is

In [24]:
```
Riem = g.riemann()
print(Riem)
```

```
Tensor field Riem(g) of type (1,3) on the 4-dimensional differentiable
manifold M
```

The component $R^0{}_{123} = R^t{}_{r\theta\phi}$ is

In [25]:
```
Riem[0,1,2,3]
```

Out[25]:

$$-\frac{\left(a^7 m - 2\,a^5 m^2 r + a^5 m r^2\right)\cos(\theta)\sin(\theta)^5 + \left(a^7 m + 2\,a^5 m^2 r + 6\,a^5 m r^2 - 6\,a^3 m^2 r^3 + 5\,a^3 m r^4\right)\cos(\theta)\sin(\theta)^3 - 2\left(a^7 m - a^5 m r^2 - 5\,a^3 m r^4 - 3\,a m r^6\right)\cos(\theta)\sin(\theta)}{a^2 r^6 - 2\,m r^7 + r^8 + \left(a^8 - 2\,a^6 m r + a^6 r^2\right)\cos(\theta)^6 + 3\left(a^6 r^2 - 2\,a^4 m r^3 + a^4 r^4\right)\cos(\theta)^4 + 3\left(a^4 r^4 - 2\,a^2 m r^5 + a^2 r^6\right)\cos(\theta)^2}$$

The Ricci tensor:

In [26]:
```
Ric = g.ricci()
print(Ric)
```

```
Field of symmetric bilinear forms Ric(g) on the 4-dimensional different
iable manifold M
```

Let us check that the Kerr metric is a vacuum solution of Einstein equation, i.e. that the Ricci tensor vanishes identically:

In [27]:
```
Ric.display()
```

Out[27]:

$$\mathrm{Ric}\left(g\right) = 0$$

The Weyl conformal curvature tensor is

In [28]:
```
C = g.weyl()
print(C)
```

```
Tensor field C(g) of type (1,3) on the 4-dimensional differentiable man
ifold M
```

Let us exhibit two of its components $C^0{}_{123}$ and $C^0{}_{101}$:

In [29]:
```
C[0,1,2,3]
```

Out[29]:

$$-\frac{\left(a^7 m - 2\,a^5 m^2 r + a^5 m r^2\right)\cos(\theta)\sin(\theta)^5 + \left(a^7 m + 2\,a^5 m^2 r + 6\,a^5 m r^2 - 6\,a^3 m^2 r^3 + 5\,a^3 m r^4\right)\cos(\theta)\sin(\theta)^3 - 2\left(a^7 m - a^5 m r^2 - 5\,a^3 m r^4 - 3\,a m r^6\right)\cos(\theta)\sin(\theta)}{a^2 r^6 - 2\,m r^7 + r^8 + \left(a^8 - 2\,a^6 m r + a^6 r^2\right)\cos(\theta)^6 + 3\left(a^6 r^2 - 2\,a^4 m r^3 + a^4 r^4\right)\cos(\theta)^4 + 3\left(a^4 r^4 - 2\,a^2 m r^5 + a^2 r^6\right)\cos(\theta)^2}$$

In [30]:
```
C[0,1,0,1]
```

Out[30]:

$$\frac{3\,a^4 mr\cos\left(\theta\right)^4 + 3\,a^2 mr^3 + 2\,mr^5 - \left(9\,a^4 mr + 7\,a^2 mr^3\right)\cos\left(\theta\right)^2}{a^2 r^6 - 2\,mr^7 + r^8 + \left(a^8 - 2\,a^6 mr + a^6 r^2\right)\cos\left(\theta\right)^6 + 3\left(a^6 r^2 - 2\,a^4 mr^3 + a^4 r^4\right)\cos\left(\theta\right)^4 + 3\left(a^4 r^4 - 2\,a^2 mr^5 + a^2 r^6\right)\cos\left(\theta\right)^2}$$

To form the Simon-Mars tensor, we need the fully covariant (type-(0,4) tensor) form of the Weyl tensor (i.e. $C_{\alpha\beta\mu\nu} = g_{\alpha\sigma} C^{\sigma}{}_{\beta\mu\nu}$); we get it by lowering the first index with the metric:

In [31]:
```
Cd = C.down(g)
print(Cd)
```

```
Tensor field of type (0,4) on the 4-dimensional differentiable manifold
 M
```

The (monoterm) symmetries of this tensor are those inherited from the Weyl tensor, i.e. the antisymmetry on the last two indices (position 2 and 3, the first index being at position 0):

In [32]:
```
Cd.symmetries()
```

```
no symmetry; antisymmetry: (2, 3)
```

Actually, Cd is also antisymmetric with respect to the first two indices (positions 0 and 1), as we can check:

In [33]:
```
Cd == Cd.antisymmetrize(0,1)
```

Out[33]: True

To take this symmetry into account explicitely, we set

In [34]:
```
Cd = Cd.antisymmetrize(0,1)
```

Hence we have now

In [35]:
```
Cd.symmetries()
```

```
no symmetry; antisymmetries: [(0, 1), (2, 3)]
```

Simon-Mars tensor

The Simon-Mars tensor with respect to the Killing vector ξ is a rank-3 tensor introduced by Marc Mars in 1999 (Class. Quantum Grav. **16**, 2507). It has the remarkable property to vanish identically if, and only if, the spacetime $(\mathcal{M}, g)$ is locally isometric to a Kerr spacetime.

Let us evaluate the Simon-Mars tensor by following the formulas given in Mars' article. The starting point is the self-dual complex 2-form associated with the Killing 2-form F, i.e. the object $\mathcal{F} := F + i\,{}^*F$, where ${}^*F$ is the Hodge dual of F:

In [36]:
```
FF = F + I * F.hodge_dual(g)
FF.set_name('FF', r'\mathcal{F}') ; print(FF)
```

```
2-form FF on the 4-dimensional differentiable manifold M
```

In [37]:
```
FF.display()
```

Out[37]:

$$\mathcal{F} = \left(-\frac{a^2 m \cos(\theta)^2 + 2i\, amr \cos(\theta) - mr^2}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4} \right) \mathrm{d}t \wedge \mathrm{d}r + \left(\frac{\left(i\, a^3 m \cos(\theta)^2 - 2\, a^2 mr \cos(\theta) - i\, amr^2\right) \sin(\theta)}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4} \right) \mathrm{d}t \wedge \mathrm{d}\theta$$

$$+ \left(\frac{\begin{array}{l} -4i\, a^4 m^2 r^2 \cos(\theta) \sin(\theta)^4 \\ + \left(a^3 mr^4 - 2\, am^2 r^5 + amr^6 - \left(a^7 m - 2\, a^5 m^2 r + a^5 mr^2\right) \cos(\theta)^4 \sin(\theta)^2 \right. \\ \quad - \left(2i\, a^6 mr + 2i\, a^4 mr^3\right) \cos(\theta)^3 \\ \quad \left. - \left(-4i\, a^4 m^2 r^2 + 2i\, a^4 mr^3 - 4i\, a^2 m^2 r^4 + 2i\, a^2 mr^5\right) \cos(\theta)\right) \end{array}}{\begin{array}{l} a^2 r^6 - 2\, mr^7 + r^8 + \left(a^8 - 2\, a^6 mr + a^6 r^2\right) \cos(\theta)^6 + 3 \\ \left(a^6 r^2 - 2\, a^4 mr^3 + a^4 r^4\right) \cos(\theta)^4 + 3\left(a^4 r^4 - 2\, a^2 mr^5 + a^2 r^6\right) \cos(\theta)^2 \end{array}} \right) \mathrm{d}$$

$$\wedge\, \mathrm{d}\phi$$

$$+ \left(-\frac{\left(i\, a^4 m + i\, a^2 mr^2\right) \sin(\theta)^3 + \left(-i\, a^4 m + i\, mr^4 + 2\left(a^3 mr + amr^3\right) \cos(\theta)\right) \sin(\theta)}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4} \right.$$

$$\wedge\, \mathrm{d}\phi$$

Let us check that $\mathcal{F}$ is self-dual, i.e. that it obeys ${}^*\mathcal{F} = -i\mathcal{F}$:

In [38]:
```
FF.hodge_dual(g) == - I * FF
```

Out[38]: True

Let us form the right self-dual of the Weyl tensor as follows

$$\mathcal{C}_{\alpha\beta\mu\nu} = C_{\alpha\beta\mu\nu} + \frac{i}{2}\epsilon^{\rho\sigma}{}_{\mu\nu}\, C_{\alpha\beta\rho\sigma},$$

where $\epsilon^{\rho\sigma}{}_{\mu\nu}$ is associated to the Levi-Civita tensor $\epsilon_{\rho\sigma\mu\nu}$ and is obtained by

In [39]:
```
eps = g.volume_form(2)  # 2 = the first 2 indices are contravariant
print(eps)
eps.symmetries()
```

```
Tensor field of type (2,2) on the 4-dimensional differentiable manifold
 M
no symmetry; antisymmetries: [(0, 1), (2, 3)]
```

The right self-dual Weyl tensor is then

In [40]:
```
CC = Cd + I/2*( eps['^rs_..']*Cd['_..rs'] )
CC.set_name('CC', r'\mathcal{C}') ; print(CC)
```

```
Tensor field CC of type (0,4) on the 4-dimensional differentiable manif
old M
```

In [41]:
```
CC.symmetries()
```

```
no symmetry; antisymmetries: [(0, 1), (2, 3)]
```

In [42]:
```
CC[0,1,2,3]
```

Out[42]:
$$\frac{\left(a^5 m \cos(\theta)^5 + 3i\, a^4 m r \cos(\theta)^4 + 3i\, a^2 m r^3 + 2i\, m r^5 - \left(3\, a^5 m + 5\, a^3 m r^2\right)\cos(\theta)^3 + \left(-9i\, a^4 m r - 7i\, a^2 m r^3\right)\cos(\theta)^2 + 3\left(3\, a^3 m r^2 + 2\, a m r^4\right)\cos(\theta)\right)\sin(\theta)}{a^6 \cos(\theta)^6 + 3\, a^4 r^2 \cos(\theta)^4 + 3\, a^2 r^4 \cos(\theta)^2 + r^6}$$

The Ernst 1-form $\sigma_\alpha = 2\mathcal{F}_{\mu\alpha}\,\xi^\mu$ (0 = contraction on the first index of $\mathcal{F}$):

In [43]:
```
sigma = 2*FF.contract(0, xi)
```

Instead of invoking the function `contract()`, we could have used the index notation to denote the contraction:

In [44]:
```
sigma == 2*( FF['_ma']*xi['^m'] )
```

Out[44]: True

In [45]:
```
sigma.set_name('sigma', r'\sigma') ; print(sigma)
sigma.display()
```

```
1-form sigma on the 4-dimensional differentiable manifold M
```

Out[45]:
$$\sigma = \left(-\frac{2\, a^2 m \cos(\theta)^2 + 4i\, a m r \cos(\theta) - 2\, m r^2}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}r + \left(\frac{\left(2i\, a^3 m \cos(\theta)^2 - 4\, a^2 m r \cos(\theta) - 2i\, a m r^2\right)\sin(\theta)}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}\theta$$

The symmetric bilinear form $\gamma = \lambda\, g + \underline{\xi} \otimes \underline{\xi}$:

In [46]:
```
gamma = lamb*g + xi_form * xi_form
gamma.set_name('gamma', r'\gamma') ; print(gamma)
gamma.display()
```

```
Field of symmetric bilinear forms gamma on the 4-dimensional differenti
able manifold M
```

Out[46]:
$$\gamma = \left(\frac{a^2 \cos(\theta)^2 - 2\, m r + r^2}{a^2 - 2\, m r + r^2}\right) \mathrm{d}r \otimes \mathrm{d}r + \left(a^2 \cos(\theta)^2 - 2\, m r + r^2\right) \mathrm{d}\theta \otimes \mathrm{d}\theta + \left(\frac{2\, a^2 m r \sin(\theta)^4 - \left(2\, a^2 m r - a^2 r^2 + 2\, m r^3 - r^4 - \left(a^4 + a^2 r^2\right)\cos(\theta)^2\right)\sin(\theta)^2}{a^2 \cos(\theta)^2 + r^2}\right) \otimes \mathrm{d}\phi$$

Final computation leading to the Simon-Mars tensor:

First we evaluate

$$S^{(1)}_{\alpha\beta\gamma} = 4 C_{\mu\alpha\nu\beta}\,\xi^\mu\,\xi^\nu\,\sigma_\gamma$$

In [47]:
```
S1 = 4*( CC.contract(0,xi).contract(1,xi) ) * sigma
print(S1)
```

```
Tensor field of type (0,3) on the 4-dimensional differentiable manifold
 M
```

Then we form the tensor

$$S^{(2)}_{\alpha\beta\gamma} = -\gamma_{\alpha\beta}\, \mathcal{C}_{\rho\gamma\mu\nu}\, \xi^\rho\, \mathcal{F}^{\mu\nu}$$

by first computing $\mathcal{C}_{\rho\gamma\mu\nu}\, \xi^\rho$:

In [48]:
```
xiCC = CC['_.r..']*xi['^r']
print(xiCC)
```

```
Tensor field of type (0,3) on the 4-dimensional differentiable manifold
 M
```

We use the index notation to perform the double contraction $\mathcal{C}_{\gamma\rho\mu\nu}\mathcal{F}^{\mu\nu}$:

In [49]:
```
FFuu = FF.up(g)
```

In [50]:
```
S2 = gamma * ( xiCC['_.mn']*FFuu['^mn'] )
print(S2)
S2.symmetries()
```

```
Tensor field of type (0,3) on the 4-dimensional differentiable manifold
 M
symmetry: (0, 1); no antisymmetry
```

The Simon-Mars tensor with respect to ξ is obtained by antisymmetrizing $S^{(1)}$ and $S^{(2)}$ on their last two indices and adding them:

$$S_{\alpha\beta\gamma} = S^{(1)}_{\alpha[\beta\gamma]} + S^{(2)}_{\alpha[\beta\gamma]}$$

We use the index notation for the antisymmetrization:

In [51]:
```
S1A = S1['_a[bc]']
S2A = S2['_a[bc]']
```

An equivalent writing would have been (the last two indices being in position 1 and 2):

In [52]:
```
# S1A = S1.antisymmetrize(1,2)
# S2A = S2.antisymmetrize(1,2)
```

The Simon-Mars tensor is

In [53]:
```
S = S1A + S2A
S.set_name('S') ; print(S)
S.symmetries()
```

```
Tensor field S of type (0,3) on the 4-dimensional differentiable manifo
ld M
no symmetry; antisymmetry: (1, 2)
```

In [54]:
```
S.display()
```

Out[54]: $S = 0$

We thus recover the fact that the Simon-Mars tensor vanishes identically in Kerr spacetime.

To check that the above computation was not trival, here is the component 112=$rr\theta$ for each of the two parts of the Simon-Mars tensor:

In [55]:
```
S1A[1,1,2]
```

Out[55]:

$$-\frac{\left(8\,a^{8}m^{2}\cos\left(\theta\right)^{7}+40i\,a^{7}m^{2}r\cos\left(\theta\right)^{6}-16i\,am^{3}r^{6}+8i\,am^{2}r^{7}-8\left(2\,a^{6}m^{3}r+9\,a^{6}m^{2}r^{2}\right)\cos\left(\theta\right)^{5}+\left(-80i\,a^{5}m^{3}r^{2}-40i\,a^{5}m^{2}r^{3}\right)\cos\left(\theta\right)^{4}+40\left(4\,a^{4}m^{3}r^{3}-a^{4}m^{2}r^{4}\right)\cos\left(\theta\right)^{3}+\left(160i\,a^{3}m^{3}r^{4}-72i\,a^{3}m^{2}r^{5}\right)\cos\left(\theta\right)^{2}-40\left(2\,a^{2}m^{3}r^{5}-a^{2}m^{2}r^{6}\right)\cos(\theta)\right)\sin\left(\theta\right)}{2\left(a^{2}r^{10}-2\,mr^{11}+r^{12}+\left(a^{12}-2\,a^{10}mr+a^{10}r^{2}\right)\cos\left(\theta\right)^{10}+5\left(a^{10}r^{2}-2\,a^{8}mr^{3}+a^{8}r^{4}\right)\cos\left(\theta\right)^{8}+10\left(a^{8}r^{4}-2\,a^{6}mr^{5}+a^{6}r^{6}\right)\cos\left(\theta\right)^{6}+10\left(a^{6}r^{6}-2\,a^{4}mr^{7}+a^{4}r^{8}\right)\cos\left(\theta\right)^{4}+5\left(a^{4}r^{8}-2\,a^{2}mr^{9}+a^{2}r^{10}\right)\cos\left(\theta\right)^{2}\right)}$$

In [56]:
```
S2A[1,1,2]
```

Out[56]:

$$\frac{\left(8\,a^{8}m^{2}\cos\left(\theta\right)^{7}+40i\,a^{7}m^{2}r\cos\left(\theta\right)^{6}-16i\,am^{3}r^{6}+8i\,am^{2}r^{7}-8\left(2\,a^{6}m^{3}r+9\,a^{6}m^{2}r^{2}\right)\cos\left(\theta\right)^{5}+\left(-80i\,a^{5}m^{3}r^{2}-40i\,a^{5}m^{2}r^{3}\right)\cos\left(\theta\right)^{4}+40\left(4\,a^{4}m^{3}r^{3}-a^{4}m^{2}r^{4}\right)\cos\left(\theta\right)^{3}+\left(160i\,a^{3}m^{3}r^{4}-72i\,a^{3}m^{2}r^{5}\right)\cos\left(\theta\right)^{2}-40\left(2\,a^{2}m^{3}r^{5}-a^{2}m^{2}r^{6}\right)\cos(\theta)\right)\sin\left(\theta\right)}{2\left(a^{2}r^{10}-2\,mr^{11}+r^{12}+\left(a^{12}-2\,a^{10}mr+a^{10}r^{2}\right)\cos\left(\theta\right)^{10}+5\left(a^{10}r^{2}-2\,a^{8}mr^{3}+a^{8}r^{4}\right)\cos\left(\theta\right)^{8}+10\left(a^{8}r^{4}-2\,a^{6}mr^{5}+a^{6}r^{6}\right)\cos\left(\theta\right)^{6}+10\left(a^{6}r^{6}-2\,a^{4}mr^{7}+a^{4}r^{8}\right)\cos\left(\theta\right)^{4}+5\left(a^{4}r^{8}-2\,a^{2}mr^{9}+a^{2}r^{10}\right)\cos\left(\theta\right)^{2}\right)}$$

In [57]:
```
S1A[1,1,2] + S2A[1,1,2]
```

Out[57]: 0