Simon-Mars tensor and Kerr spacetime

This worksheet demonstrates a few capabilities of SageManifolds (version 1.0, as included in SageMath 7.5) regarding the computation of the Simon-Mars tensor.

Click here to download the worksheet file (ipynb format). To run it, you must start SageMath within the Jupyter notebook, via the command sage -n jupyter

NB: a version of SageMath at least equal to 7.5 is required to run this worksheet:

In [1]: version()
Out[1]: 'SageMath version 7.5.1, Release Date: 2017-01-15'

First we set up the notebook to display mathematical objects using LaTeX rendering:

In [2]: %display latex

Since some computations are quite long, we ask for running them in parallel on 8 cores:

In [3]: Parallelism().set(nproc=8)

Spacetime manifold

We declare the Kerr spacetime (or more precisely the part of the Kerr spacetime covered by Boyer-Lindquist coordinates) as a 4-dimensional manifold \( \mathcal{M} \):

In [4]: M = Manifold(4, 'M', latex_name=r'\mathcal{M}')
   print(M)

4-dimensional differentiable manifold M

The standard Boyer-Lindquist coordinates \((t, r, \theta, \phi)\) are introduced by declaring a chart \(X\) on \(\mathcal{M}\) via the method chart(), the argument of which is a string expressing the coordinates names, their ranges (the default is \((-\infty, +\infty)\)) and their LaTeX symbols:

In [5]: X.<t,r,th,ph> = M.chart(r't:(0,+oo) r:(0,2*pi) \theta:(0,pi) \phi:(0,2*pi)')
   print(X) ; X

Chart \((\mathcal{M}, (t, r, \theta, \phi))\)

Out[5]: \((\mathcal{M}, (t, r, \theta, \phi))\)

Metric tensor

The 2 parameters \(m\) and \(a\) of the Kerr spacetime are declared as symbolic variables:

In [6]: var('m, a', domain='real')
Out[6]: \((m, a)\)
Let us introduce the spacetime metric $g$ and set its components in the coordinate frame associated with Boyer-Lindquist coordinates, which is the current manifold’s default frame:

$$g = M.lorentzian\text{metric}('g')$$

$$g[0,0] = r^2 - 2m^2r + a^2$$

$$g[0,3] = -2a^2m^2r^2\sin^2(\theta)^2$$

$$g[1,1] = r^2$$

$$g[2,2] = \frac{r^2}{\Delta}$$

$$g[3,3] = \frac{(r^2 + a^2)(r^2 + 2am^2r^2(\sin^2(\theta))^2)}{\Delta}$$

In [7]:

$$\nabla_g g = \frac{2mr}{a^2 \cos^2(\theta)^2 + r^2} dr \otimes dt + \frac{2amr \sin(\theta)^2}{a^2 \cos^2(\theta)^2 + r^2} dr \otimes d\phi$$

$$+ \frac{a^2 \cos(\theta)^2 + r^2}{a^2 - 2mr + r^2} dr \otimes dr + \frac{2a^2m \sin(\theta)^2}{a^2 \cos^2(\theta)^2 + a^2 + r^2} \sin(\theta)^2 d\phi \otimes d\phi$$

Out[7]:

$$\begin{pmatrix}
\frac{2mr}{a^2 \cos^2(\theta)^2 + r^2} - 1 & 0 & 0 & -\frac{2amr \sin(\theta)^2}{a^2 \cos^2(\theta)^2 + r^2} \\
0 & \frac{a^2 \cos^2(\theta)^2 + r^2}{a^2 - 2mr + r^2} & 0 & 0 \\
0 & 0 & a^2 \cos^2(\theta)^2 + r^2 & 0 \\
-\frac{2amr \sin(\theta)^2}{a^2 \cos^2(\theta)^2 + r^2} & 0 & 0 & \frac{2a^2m \sin(\theta)^2}{a^2 \cos^2(\theta)^2 + a^2 + r^2} \sin(\theta)^2
\end{pmatrix}$$

In [8]:

$$\nabla_g g$$

Out[8]:

The Levi-Civita connection $\nabla$ associated with $g$:

In [9]:

$$\nabla = g\text{.connection}()$$

Levi-Civita connection $\nabla_g g$ associated with the Lorentzian metric $g$

As a check, we verify that the covariant derivative of $g$ with respect to $\nabla$ vanishes identically:

In [10]:

$$\nabla_g g\text{.display}()$$

Out[10]:

$$\nabla_g g = 0$$

**Killing vector**

The default vector frame on the spacetime manifold is the coordinate basis associated with Boyer-Lindquist coordinates:
In [11]: M.default_frame() is X.frame()
Out[11]: True

In [12]: X.frame()
Out[12]: (\mathcal{M},\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\right))

Let us consider the first vector field of this frame:

In [13]: xi = X.frame()[0]; xi
Out[13]: \frac{\partial}{\partial t}

In [14]: print(xi)
Vector field d/dt on the 4-dimensional differentiable manifold M

The 1-form associated to it by metric duality is

In [15]: xi_form = xi.down(g)
xi_form.set_name('xi_form', r'\underline{\xi}')
print(xi_form); xi_form.display()

1-form xi_form on the 4-dimensional differentiable manifold M

\[ \xi = \left( -\frac{a^2 \cos(\theta)^2 - 2mr + r^2}{a^2 \cos(\theta)^2 + r^2} \right) dt + \left( -\frac{2amr \sin(\theta)^2}{a^2 \cos(\theta)^2 + r^2} \right) d\phi \]

Its covariant derivative is
Tensor field nabla g(xi_form) of type (0,2) on the 4-dimensional differentiable manifold M

\[
\nabla_{\xi} = \frac{a^2 m \cos (\theta)^2 - mr^2}{a^4 \cos (\theta)^4 + 2 a^2 r^2 \cos (\theta)^2 + r^4} dr \otimes dr \\
+ \frac{2 a^2 m \cos (\theta)^2 - mr^2}{a^4 \cos (\theta)^4 + 2 a^2 r^2 \cos (\theta)^2 + r^4} dr \otimes d\theta \\
+ \frac{a^3 m \cos (\theta)^2 - amr^2}{a^4 \cos (\theta)^4 + 2 a^2 r^2 \cos (\theta)^2 + r^4} d\theta \otimes dr \\
+ \frac{2 a^2 m \cos (\theta)^2}{a^4 \cos (\theta)^4 + 2 a^2 r^2 \cos (\theta)^2 + r^4} d\theta \otimes d\phi \\
+ \frac{2 (a^3 m r + amr^3) \cos (\theta) \sin (\theta)}{a^4 \cos (\theta)^4 + 2 a^2 r^2 \cos (\theta)^2 + r^4} d\phi \otimes d\phi \\
+ \frac{a^3 m \cos (\theta)^2 - amr^2}{a^4 \cos (\theta)^4 + 2 a^2 r^2 \cos (\theta)^2 + r^4} d\phi \otimes d\theta \\
+ \frac{2 (a^3 m r + amr^3) \cos (\theta) \sin (\theta)}{a^4 \cos (\theta)^4 + 2 a^2 r^2 \cos (\theta)^2 + r^4} d\phi \otimes d\theta
\]

Let us check that the Killing equation is satisfied:

Equivalently, we check that the Lie derivative of the metric along \( \xi \) vanishes:

Thank to Killing equation, \( \nabla_{\xi} g \) is antisymmetric. We may therefore define a 2-form by \( F := -\nabla_{\xi} g \). Here we enforce the antisymmetry by calling the function \( \text{antisymmetrize}() \) on \( \nabla_{\xi} g \).
In [19]:

    F = - nab_xi.antisymmetrize()
    F.set_name('F')
    print(F)
    F.display()

2-form $F$ on the 4-dimensional differentiable manifold $M$

\[
F = \left( -\frac{a^2 m \cos(\theta)^2 - m r^2}{a^4 \cos(\theta)^4 + 2 a^2 r^2 \cos(\theta)^2 + r^4} \right) dr \wedge dr
+ \left( -\frac{2 a^2 m r \cos(\theta) \sin(\theta)}{a^4 \cos(\theta)^4 + 2 a^2 r^2 \cos(\theta)^2 + r^4} \right) dr \wedge d\theta
+ \left( -\frac{a^3 m \cos(\theta)^2 - a m r^2}{a^4 \cos(\theta)^4 + 2 a^2 r^2 \cos(\theta)^2 + r^4} \right) dr \wedge d\phi
+ \left( -\frac{2 (a^3 m r + a m^3) \cos(\theta) \sin(\theta)}{a^4 \cos(\theta)^4 + 2 a^2 r^2 \cos(\theta)^2 + r^4} \right) d\theta \wedge d\phi
\]

We check that

In [20]:

    F == - nab_xi

Out[20]:

    True

The squared norm of the Killing vector is:

In [21]:

    lamb = - g(xi,xi)
    lamb.set_name('lambda', r'\lambda')
    print(lamb)
    lamb.display()

Scalar field $\lambda$ on the 4-dimensional differentiable manifold $M$

\[
\lambda : \mathcal{M} \rightarrow \mathbb{R}
(t, r, \theta, \phi) \mapsto \frac{a^2 \cos(\theta)^2 - 2 m r + r^2}{a^2 \cos(\theta)^4 + r^2}
\]

Instead of invoking $g(\xi, \xi)$, we could have evaluated $\lambda$ by means of the 1-form $\xi$ acting on the vector field $\xi$.

In [22]:

    lamb == - xi_form(xi)

Out[22]:

    True

or, using index notation as $\lambda = -\xi^a \xi_a$:

In [23]:

    lamb == - ( xi_form['a']*xi['^a'] )

Out[23]:

    True

Curvature

The Riemann curvature tensor associated with $g$ is
The component $R_{123}^0 = R^r_{\theta \phi}$ is

\[
\begin{align*}
(a^7 m - 2 a^5 m^2 r + a^5 m r^2) \cos(\theta) \sin(\theta)^5 \\
+ (a^7 m + 2 a^5 m^2 r + 6 a^4 m r^2 - 6 a^3 m^2 r^3 + 5 a^3 m r^4) \cos(\theta) \sin(\theta)^3 - 2 \\
(a^7 m - a^5 m r^2 - 5 a^3 m r^4 - 3 a m r^6) \cos(\theta) \sin(\theta) \\
\frac{a^2 r^6 - 2 m r^7 + r^8 + (a^8 - 2 a^6 m r + a^6 r^2) \cos(\theta)^6 + 3}{a^2 r^6 - 2 a^4 m r^3 + a^4 r^4} \cos(\theta)^4 + 3 (a^4 r^4 - 2 a^2 m r^5 + a^2 r^6) \cos(\theta)^2
\end{align*}
\]

The Ricci tensor:

\[
\text{Ric} = g.\text{ricci()}
\]

Field of symmetric bilinear forms $\text{Ric}(g)$ on the 4-dimensional differentiable manifold $M$

Let us check that the Kerr metric is a vacuum solution of Einstein equation, i.e. that the Ricci tensor vanishes identically:

\[
\text{Ric}.\text{display()}
\]

The Weyl conformal curvature tensor is

\[
\text{C} = g.\text{weyl()}
\]

Tensor field $\text{C}(g)$ of type $(1,3)$ on the 4-dimensional differentiable manifold $M$

Let us exhibit two of its components $C_{123}^0$ and $C_{101}^0$:

\[
\begin{align*}
(a^7 m - 2 a^5 m^2 r + a^5 m r^2) \cos(\theta) \sin(\theta)^5 \\
+ (a^7 m + 2 a^5 m^2 r + 6 a^4 m r^2 - 6 a^3 m^2 r^3 + 5 a^3 m r^4) \cos(\theta) \sin(\theta)^3 - 2 \\
(a^7 m - a^5 m r^2 - 5 a^3 m r^4 - 3 a m r^6) \cos(\theta) \sin(\theta) \\
\frac{a^2 r^6 - 2 m r^7 + r^8 + (a^8 - 2 a^6 m r + a^6 r^2) \cos(\theta)^6 + 3}{a^2 r^6 - 2 a^4 m r^3 + a^4 r^4} \cos(\theta)^4 + 3 (a^4 r^4 - 2 a^2 m r^5 + a^2 r^6) \cos(\theta)^2
\end{align*}
\]
In [30]: C[0,1,0,1]

Out[30]:
\[
\begin{align*}
3a^4mr \cos(\theta)^4 + 3a^2mr^3 + 2mr^5 - \left(9a^4mr + 7a^2mr^3\right)\cos(\theta)^2 \\
\frac{a^2r^6 - 2mr^7 + r^8 + \left(a^8 - 2a^6mr + a^6r^2\right)\cos(\theta)^6 + 3}{(a^6r^2 - 2a^4mr^4 + a^4r^6)\cos(\theta)^4 + 3}
\end{align*}
\]

To form the Simon-Mars tensor, we need the fully covariant (type-(0,4) tensor) form of the Weyl tensor (i.e. \( C_{\alpha\beta\mu\nu} = g_{\alpha\sigma}C^\sigma_{\beta\mu\nu} \)); we get it by lowering the first index with the metric:

In [31]: Cd = C.down(g)

print(Cd)

Tensor field of type (0,4) on the 4-dimensional differentiable manifold \( M \)

The (monoterm) symmetries of this tensor are those inherited from the Weyl tensor, i.e. the antisymmetry on the last two indices (position 2 and 3, the first index being at position 0):

In [32]: Cd.symmetries()

no symmetry; antisymmetry: (2, 3)

Actually, Cd is also antisymmetric with respect to the first two indices (positions 0 and 1), as we can check:

In [33]: Cd == Cd.antisymmetrize(0,1)

Out[33]: True

To take this symmetry into account explicitly, we set

In [34]: Cd = Cd.antisymmetrize(0,1)

Hence we have now

In [35]: Cd.symmetries()

no symmetry; antisymmetries: [(0, 1), (2, 3)]

**Simon-Mars tensor**

The Simon-Mars tensor with respect to the Killing vector \( \xi \) is a rank-3 tensor introduced by Marc Mars in 1999 (Class. Quantum Grav. 16, 2507). It has the remarkable property to vanish identically if, and only if, the spacetime \( (\mathcal{M}, g) \) is locally isometric to a Kerr spacetime.

Let us evaluate the Simon-Mars tensor by following the formulas given in Mars' article. The starting point is the self-dual complex 2-form associated with the Killing 2-form \( F \), i.e. the object \( F' := F + i^*F \), where \( ^*F \) is the Hodge dual of \( F \):

In [36]: FF = F + I * F.hodge_dual(g)

FF.set_name('FF', r'\mathcal{F}'); print(FF)

2-form FF on the 4-dimensional differentiable manifold \( M \)
In [41]:

Out[41]:

no symmetry; antisymmetries: \[(0, 1), (2, 3)\]

old \text{M}^\text{Tensor field CC of type (0,4) on the 4-dimensional differentiable mani}

where

Let us check that \( \mathcal{F} \) is self-dual, i.e. that it obeys *\( \mathcal{F} = -i \mathcal{F} \* *

In [38]:

Out[38]:

True

Let us form the right self-dual of the Weyl tensor as follows

\[
C_{\alpha\beta\mu\nu} = C_{\alpha\beta\mu\nu} + \frac{i}{2} \varepsilon_{\mu\nu} C_{\alpha\beta\mu\nu},
\]

where \( \varepsilon_{\mu\nu} \) is associated to the Levi-Civita tensor \( \varepsilon_{\mu\nu} \) and is obtained by

In [39]:

Tensor field of type (2,2) on the 4-dimensional differentiable manifold \( \text{M} \)
no symmetry; antisymmetries: \([0, 1], (2, 3)\]

The right self-dual Weyl tensor is then

In [40]:

Tensor field CC of type (0,4) on the 4-dimensional differentiable manifold \( \text{M} \)

In [41]:

no symmetry; antisymmetries: \([0, 1], (2, 3)\]
First we evaluate the Ernst 1-form \( \sigma = 2F_{\mu \nu} \frac{\partial}{\partial \nu} \) (\( 0 = \) contraction on the first index of \( F \)):

\[
\sigma = 2*\mathrm{FF}.\text{contract}(\theta, \xi)
\]

Instead of invoking the function \texttt{contract()}, we could have used the index notation to denote the contraction:

\[
\sigma == 2*( \texttt{FF['\_\_m'])*xi['\_m']} )
\]

\[
\text{True}
\]

\[
\text{sigma.set_name('sigma', r'	extbackslash'sigma'); print(sigma)}
\]

1-form \( \sigma \) on the 4-dimensional differentiable manifold \( \mathbb{M} \)

\[
\sigma = \left( \frac{-2a^2m \cos(\theta)^2 + 4i amr \cos(\theta) - 2mr^2}{a^4 \cos(\theta)^4 + 2a^2r^2 \cos(\theta)^2 + r^4} \right) d\theta
\]

\[
\text{The symmetric bilinear form } \gamma = \lambda g + \xi \otimes \xi:
\]

\[
\text{gamma = lamb}\star g + \xi \text{ form } \times \text{ xi form}
\]

Field of symmetric bilinear forms \( \gamma \) on the 4-dimensional differentiable manifold \( \mathbb{M} \)

\[
\gamma = \left( \frac{a^2 \cos(\theta)^2 - 2mr + r^2}{a^2 - 2mr + r^2} \right) d\theta \otimes d\theta
\]

\[
\text{Final computation leading to the Simon-Mars tensor:}
\]

First we evaluate

\[
S_{\alpha\beta}^{(1)} = 4C_{\mu \nu \alpha} \frac{\partial}{\partial \nu} \xi^\mu \xi^\nu \sigma_\nu
\]

\[
\text{S1 = 4*( CC.contract(0,xi).contract(1,xi) ) * sigma}
\]

Tensor field of type \((0,3)\) on the 4-dimensional differentiable manifold \( \mathbb{M} \)
Then we form the tensor

\[ S_{\alpha\beta\gamma}^{(2)} = -\gamma_{\alpha\beta\gamma} \xi^\rho F^{\mu\nu} \]

by first computing \( C_{\rho\mu\nu} \xi^\rho \):

In [48]:

```python
xiCC = CC[\'__r..\']^xi[\'^r\']
print(xiCC)
```

Tensor field of type (0,3) on the 4-dimensional differentiable manifold \( M \)

We use the index notation to perform the double contraction \( C_{\rho\mu\nu} F^{\mu\nu} \):

In [49]:

```python
FFuu = FF.up(g)
```

In [50]:

```python
S2 = gamma * ( xiCC[\'__mn\']^FFuu[\'^mn\'] )
print(S2)
S2.symmetries()
```

Tensor field of type (0,3) on the 4-dimensional differentiable manifold \( M \)
symmetry: (0, 1); no antisymmetry

The Simon-Mars tensor with respect to \( \xi \) is obtained by antisymmetrizing \( S^{(1)} \) and \( S^{(2)} \) on their last two indices and adding them:

\[ S_{\alpha\beta\gamma} = S_{\alpha\beta\gamma}^{(1)} + S_{\alpha\beta\gamma}^{(2)} \]

We use the index notation for the antisymmetrization:

In [51]:

```python
S1A = S1[\'__a[bc]\']
S2A = S2[\'__a[bc]\']
```

An equivalent writing would have been (the last two indices being in position 1 and 2):

In [52]:

```python
# S1A = S1.antisymmetrize(1,2)
# S2A = S2.antisymmetrize(1,2)
```

The Simon-Mars tensor is

In [53]:

```python
S = S1A + S2A
S.set_name('S') ; print(S)
S.symmetries()
```

Tensor field \( S \) of type (0,3) on the 4-dimensional differentiable manifold \( M \)
no symmetry; antisymmetry: (1, 2)

In [54]:

```python
S.display()
```

Out[54]:

\[ S = 0 \]

We thus recover the fact that the Simon-Mars tensor vanishes identically in Kerr spacetime.

To check that the above computation was not trivial, here is the component \( 112=rr\theta \) for each of the two parts of the Simon-Mars tensor:
\[\begin{align*}
\text{Out}[55]: & \quad (8a^8m^2 \cos(\theta)^7 + 40i a^7 m^2 r \cos(\theta)^6 - 16i a m^3 r^6 + 8i a m^2 r^7 - 8 \sin(\theta) \\
& \quad (2a^6m^3 r + 9a^6m^2 r^2) \cos(\theta)^5 + (-80i a^5 m^3 r^2 - 40i a^5 m^2 r^3) \cos(\theta)^4 + 40 \\
& \quad (4a^4m^3 r^3 - a^4 m^2 r^4) \cos(\theta)^3 + (160i a^3 m^3 r^4 - 72i a^3 m^2 r^5) \cos(\theta)^2 - 40 \\
& \quad (2a^2 m^3 r^5 - a^2 m^2 r^6) \cos(\theta) \\
& \quad - \frac{2}{a^2 r^{10} - 2m r^{11} + r^{12} + (a^{12} - 2a^{10} m r + a^{10} r^2) \cos(\theta)^{10} + 5} \\
& \quad (a^{10} r^2 - 2a^8 m r^3 + a^8 r^4) \cos(\theta)^8 + 10 (a^8 m r^4 + a^6 r^6) \cos(\theta)^6 + 10 \\
& \quad (a^8 r^6 - 2a^4 m r^7 + a^4 r^8) \cos(\theta)^4 + 5 (a^4 r^8 - 2a^2 m r^9 + a^2 r^{10}) \cos(\theta)^2) \\
\text{Out}[56]: & \quad (8a^8m^2 \cos(\theta)^7 + 40i a^7 m^2 r \cos(\theta)^6 - 16i a m^3 r^6 + 8i a m^2 r^7 - 8 \sin(\theta) \\
& \quad (2a^6m^3 r + 9a^6m^2 r^2) \cos(\theta)^5 + (-80i a^5 m^3 r^2 - 40i a^5 m^2 r^3) \cos(\theta)^4 + 40 \\
& \quad (4a^4m^3 r^3 - a^4 m^2 r^4) \cos(\theta)^3 + (160i a^3 m^3 r^4 - 72i a^3 m^2 r^5) \cos(\theta)^2 - 40 \\
& \quad (2a^2 m^3 r^5 - a^2 m^2 r^6) \cos(\theta) \\
& \quad - \frac{2}{a^2 r^{10} - 2m r^{11} + r^{12} + (a^{12} - 2a^{10} m r + a^{10} r^2) \cos(\theta)^{10} + 5} \\
& \quad (a^{10} r^2 - 2a^8 m r^3 + a^8 r^4) \cos(\theta)^8 + 10 (a^8 m r^4 + a^6 r^6) \cos(\theta)^6 + 10 \\
& \quad (a^8 r^6 - 2a^4 m r^7 + a^4 r^8) \cos(\theta)^4 + 5 (a^4 r^8 - 2a^2 m r^9 + a^2 r^{10}) \cos(\theta)^2) \\
\text{Out}[57]: & \quad 0
\end{align*}\]