3+1 Einstein equations in the $\delta = 2$ Tomimatsu-Sato spacetime

This worksheet demonstrates a few capabilities of SageManifolds (version 1.0, as included in SageMath 7.5) in computations regarding the 3+1 slicing of the $\delta = 2$ Tomimatsu-Sato spacetime.

Click here to download the worksheet file (ipynb format). To run it, you must start SageMath with the Jupyter notebook, via the command `sage -n jupyter`

NB: a version of SageMath at least equal to 7.5 is required to run this worksheet:

In [1]:
```
version()
```

Out[1]: 'SageMath version 7.5.1, Release Date: 2017-01-15'

First we set up the notebook to display mathematical objects using LaTeX rendering:

In [2]:
```
%display latex
```

Since some computations are quite long, we ask for running them in parallel on 8 cores:

In [3]:
```
Parallelism().set(nproc=8)
```

Tomimatsu-Sato spacetime

The Tomimatsu-Sato solution is an exact stationary and axisymmetric solution of the vaccum Einstein equation, which is asymptotically flat and has a non-zero angular momentum. It has been found in 1972 by A. Tomimatsu and H. Sato [Phys. Rev. Lett. **29**, 1344 (1972)], as a solution of the Ernst equation. It is actually the member $\delta = 2$ of a larger family of solutions parametrized by a positive integer δ and exhibited by Tomimatsu and Sato in 1973 [Prog. Theor. Phys. **50**, 95 (1973)], the member $\delta = 1$ being nothing but the Kerr metric. We refer to [Manko, Prog. Theor. Phys. **127**, 1057 (2012)] for a discussion of the properties of this solution.

Spacelike hypersurface

We consider some hypersurface Σ of a spacelike foliation $(\Sigma_t)_{t\in\mathbb{R}}$ of $\delta = 2$ Tomimatsu-Sato spacetime; we declare Σ_t as a 3-dimensional manifold:

In [4]:
```
Sig = Manifold(3, 'Sigma', r'\Sigma', start_index=1)
```

On Σ, we consider the prolate spheroidal coordinates (x, y, ϕ), with $x \in (1, +\infty)$, $y \in (-1, 1)$ and $\phi \in (0, 2\pi)$:

In [5]:
```
X.<r,y,ph> = Sig.chart(r'x:(1,+oo) y:(-1,1) ph:(0,2*pi):\phi')
print(X) ; X
```

Chart (Sigma, (x, y, ph))

Out[5]: $(\Sigma, (x, y, \phi))$

Riemannian metric on Σ

The Tomimatsu-Sato metric depens on three parameters: the integer δ, the real number $p \in [0, 1]$, and the total mass m:

In [6]:
```
var('d, p, m')
assume(m>0)
assumptions()
```

Out[6]:

$$\left[\mathtt{x\ is\ real}, x > 1, \mathtt{y\ is\ real}, y > (-1), y < 1, \mathtt{ph\ is\ real}, \phi > 0, \phi < 2\,\pi, m > 0\right]$$

We set $\delta = 2$ and choose a specific value for $p = 1/5$:

In [7]:
```
d = 2
p = 1/5
```

Furthermore, without any loss of generality, we may set $m = 1$ (this simply fixes some length scale):

In [8]:
```
m = 1
```

The parameter q is related to p by $p^2 + q^2 = 1$:

In [9]:
```
q = sqrt(1-p^2)
```

Some shortcut notations:

In [10]:
```
AA2 = (p^2*(x^2-1)^2+q^2*(1-y^2)^2)^2 \
      - 4*p^2*q^2*(x^2-1)*(1-y^2)*(x^2-y^2)^2
BB2 = (p^2*x^4+2*p*x^3-2*p*x+q^2*y^4-1)^2 \
      + 4*q^2*y^2*(p*x^3-p*x*y^2-y^2+1)^2
CC2 = p^3*x*(1-x^2)*(2*(x^4-1)+(x^2+3)*(1-y^2)) \
      + p^2*(x^2-1)*((x^2-1)*(1-y^2)-4*x^2*(x^2-y^2)) \
      + q^2*(1-y^2)^3*(p*x+1)
```

The Riemannian metric γ induced by the spacetime metric g on Σ:

In [11]:
```
gam = Sig.riemannian_metric('gam', latex_name=r'\gamma')
gam[1,1] = m^2*BB2/(p^2*d^2*(x^2-1)*(x^2-y^2)^3)
gam[2,2] = m^2*BB2/(p^2*d^2*(y^2-1)*(-x^2+y^2)^3)
gam[3,3] = - m^2*(y^2-1)*(p^2*BB2^2*(x^2-1)
                          + 4*q^2*d^2*CC2^2*(y^2-1))/(AA2*BB2*d^2)
gam.display()
```

Out[11]:

$$\gamma = \left(\frac{96\left(x^3 - xy^2 - 5\,y^2 + 5\right)^2 y^2 + \left(x^4 + 24\,y^4 \cdots\right.}{100\left(x^2 - y^2\right)^3 \left(x^2 - 1 \cdots\right.} \right.$$

$$\otimes \mathrm{d}x + \left(-\frac{96\left(x^3 - xy^2 - 5\,y^2 + 5\right)^2 y^2 + \left(x^4 + 24\,y \cdots\right.}{100\left(x^2 - y^2\right)^3 \left(y^2 - \cdots\right.} \right.$$

$$\otimes \mathrm{d}y$$

$$\left(\left(96\left(x^3 - xy^2 - 5\,y^2 + 5\right)^2 y^2 + \left(x^4 + 24\,y^4 + 10\,x^3 - 10 \cdots\right.\right.\right.$$

$$+\,9600$$

$$-\frac{\left(24\left(y^2-1\right)^3(x+5) + \left(2\,x^4 - \left(x^2+3\right)\left(y^2-1\right) - 2\right)\left(x^2 \cdots\right.\right. \quad \left(4\left(x^2-y^2\right)x^2 + \left(x^2-1\right)\left(y^2-1\right)\right)\left(x^2-1\right)\Big)}{100 \left(96\left(x^2-y^2\right)^2\left(x^2-1\right)\left(y^2-1\right) + \left(\left(x^2-1\right)^2 + 24\left(y^2-1\right)^2\right)^2\right)\left(96\left(x^3 - xy^2 \cdots\right.\right.}$$

A view of the non-vanishing components of γ w.r.t. coordinates (x, y, ϕ):

In [12]:
```
gam.display_comp()
```

Out[12]:

$$\gamma_{\,x\,x} = \frac{96\left(x^3-xy^2-5\,y^2+5\right)^2 y^2+\left(x^4+24\,y^4+10\,x^3-10\,x-25\right)^2}{100\left(x^2-y^2\right)^3\left(x^2-1\right)}$$

$$\gamma_{\,y\,y} = -\frac{96\left(x^3-xy^2-5\,y^2+5\right)^2 y^2+\left(x^4+24\,y^4+10\,x^3-10\,x-25\right)^2}{100\left(x^2-y^2\right)^3\left(y^2-1\right)}$$

$$\gamma_{\,\phi\,\phi} = -\frac{\left(\left(96\left(x^3-xy^2-5\,y^2+5\right)^2y^2+\left(x^4+24\,y^4+10\,x^3-10\,x-25\right)^2\right)^2\left(x^2-1\right)+9600 \left(24\left(y^2-1\right)^3(x+5)+\left(2\,x^4-\left(x^2+3\right)\left(y^2-1\right)-2\right)\left(x^2-1\right)x+5\left(4\left(x^2-y^2\right)x^2+\left(x^2-1\right)\left(y^2-1\right)\right)\left(x^2-1\right)\right)}{100\left(96\left(x^2-y^2\right)^2\left(x^2-1\right)\left(y^2-1\right)+\left(\left(x^2-1\right)^2+24\left(y^2-1\right)^2\right)^2\right)\left(96\left(x^3-xy^2-5\,y^2+5\right)^2y^2+\left(x^4+24\,y^4+1 \cdots\right.\right.}$$

Lapse function and shift vector

In [13]:

```
N2 = AA2/BB2 - 2*m*q*CC2*(y^2-1)/BB2*(2*m*q*CC2*(y^2-1)
               /(BB2*(m^2*(y^2-1)*(p^2*BB2^2*(x^2-1)
               +4*q^2*d^2*CC2^2*(y^2-1))/(AA2*BB2*d^2))))
N2.simplify_full()
```

Out[13]:

$$\frac{x^{10} + 20\,x^{9} + 576\,\left(x^{2} - 1\right)y^{8} + 99\,x^{8} - 40\,x^{7} + 96\,\left(x^{4} + 10\,x^{3} + 24\,x^{2} - 10\,x - 25\right)y^{6} - 350\,x^{6} - 480\,x^{5} - 48\,\left(3\,x^{6} + 10\,x^{5} - 3\,x^{4} + 20\,x^{3} + 125\,x^{2} - 30\,x - 125\right)y^{4} + 350\,x^{4} + 1000\,x^{3} + 96\,\left(x^{8} - x^{6} + 10\,x^{5} - 10\,x^{3} + 25\,x^{2} - 25\right)y^{2} + 525\,x^{2} - 500\,x - 625}{x^{10} + 40\,x^{9} + 576\,\left(x^{2} - 1\right)y^{8} + 699\,x^{8} + 7920\,x^{7} + 96\,\left(x^{4} + 20\,x^{3} + 174\,x^{2} + 980\,x + 2425\right)y^{6} + 39450\,x^{6} - 960\,x^{5} - 48\,\left(3\,x^{6} + 20\,x^{5} - 3\,x^{4} + 40\,x^{3} + 925\,x^{2} + 5940\,x + 14675\right)y^{4} - 39450\,x^{4} - 6000\,x^{3} + 96\,\left(x^{8} - x^{6} + 20\,x^{5} - 20\,x^{3} + 375\,x^{2} + 3000\,x + 7425\right)y^{2} - 9675\,x^{2} - 97000\,x - 240625}$$

In [14]:

```
N = Sig.scalar_field(sqrt(N2.simplify_full()), name='N')
print(N)
N.display()
```

```
Scalar field N on the 3-dimensional differentiable manifold Sigma
```

Out[14]:

$$\begin{array}{llcl} N: & \Sigma & \longrightarrow & \mathbb{R} \\ & (x, y, \phi) & \longmapsto & \sqrt{\frac{x^{10}+20\,x^{9}+576\,\left(x^{2}-1\right)y^{8}+99\,x^{8}-40\,x^{7}+96\,\left(x^{4}+10\,x^{3}+24\,x^{2}-10\,x-25\right)y^{6}-350 \ldots \left(3\,x^{6}+10\,x^{5}-3\,x^{4}+20\,x^{3}+125\,x^{2}-30\,x-125\right)y^{4}+350\,x^{4}+1000\,x^{3}+96\,\left(x^{8}-x^{6}+1\ldots +525\,x^{2}-500\,x-625}{x^{10}+40\,x^{9}+576\,\left(x^{2}-1\right)y^{8}+699\,x^{8}+7920\,x^{7}+96\,\left(x^{4}+20\,x^{3}+174\,x^{2}+980\,x+2425\right) \left(3\,x^{6}+20\,x^{5}-3\,x^{4}+40\,x^{3}+925\,x^{2}+5940\,x+14675\right)y^{4}-39450\,x^{4}-6000\,x^{3}+96 \left(x^{8}-x^{6}+20\,x^{5}-20\,x^{3}+375\,x^{2}+3000\,x+7425\right)y^{2}-9675\,x^{2}-97000\,x-240625}} \end{array}$$

The coordinate expression of the scalar field N:

In [15]:

```
N.expr()
```

Out[15]:

$$\sqrt{\frac{x^{10} + 20\,x^{9} + 576\,\left(x^{2} - 1\right)y^{8} + 99\,x^{8} - 40\,x^{7} + 96\,\left(x^{4} + 10\,x^{3} + 24\,x^{2} - 10\,x - 25\right)y^{6} - 350\,x^{6} - 480\,x^{5} - 48\,\left(3\,x^{6} + 10\,x^{5} - 3\,x^{4} + 20\,x^{3} + 125\,x^{2} - 30\,x - 125\right)y^{4} + 350\,x^{4} + 1000\,x^{3} + 96\,\left(x^{8} - x^{6} + 10\,x^{5} - 10\,x^{3} + 25\,x^{2} - 25\right)y^{2} + 525\,x^{2} - 500\,x - 625}{x^{10} + 40\,x^{9} + 576\,\left(x^{2} - 1\right)y^{8} + 699\,x^{8} + 7920\,x^{7} + 96\,\left(x^{4} + 20\,x^{3} + 174\,x^{2} + 980\,x + 2425\right)y^{6} + 39450\,x^{6} - 960\,x^{5} - 48\,\left(3\,x^{6} + 20\,x^{5} - 3\,x^{4} + 40\,x^{3} + 925\,x^{2} + 5940\,x + 14675\right)y^{4} - 39450\,x^{4} - 6000\,x^{3} + 96\,\left(x^{8} - x^{6} + 20\,x^{5} - 20\,x^{3} + 375\,x^{2} + 3000\,x + 7425\right)y^{2} - 9675\,x^{2} - 97000\,x - 240625}}$$

In [16]:
```
b3 = 2*m*q*CC2*(y^2-1)/(BB2*(m^2*(y^2-1)*(p^2*BB2^2*(x^2-1)
        +4*q^2*d^2*CC2^2*(y^2-1))/(AA2*BB2*d^2)))
b = Sig.vector_field('beta', latex_name=r'\beta')
b[3] = b3.simplify_full()
# unset components are zero
b.display_comp(only_nonzero=False)
```

Out[16]:

$$\beta^x = 0$$

$$\beta^y = 0$$

$$\beta^\phi = -\frac{400\left(2\sqrt{6}x^7+24\left(\sqrt{6}x+5\sqrt{6}\right)y^6+20\sqrt{6}x^6-\sqrt{6}x^5-72\left(\sqrt{6}x+5\sqrt{6}\right)y^4-25\sqrt{6}x^4-\left(\sqrt{6}x^5+15\sqrt{6}x^4+2\sqrt{6}x^3-10\sqrt{6}x^2-75\sqrt{6}x-365\sqrt{6}\right)y^2+10\sqrt{6}x^2-25\sqrt{6}x-125\sqrt{6}\right)}{x^{10}+40x^9+576\left(x^2-1\right)y^8+699x^8+7920x^7+96\left(x^4+20x^3+174x^2+980x+2425\right)y^6+39450x^6-960\ldots}$$

$$\left(3x^6+20x^5-3x^4+40x^3+925x^2+5940x+14675\right)y^4-39450x^4-6000x^3+96$$

$$\left(x^8-x^6+20x^5-20x^3+375x^2+3000x+7425\right)y^2-9675x^2-97000x-240625$$

Extrinsic curvature of Σ

We use the formula

$$K_{ij} = \frac{1}{2N}\mathcal{L}_\beta \gamma_{ij},$$

which is valid for any stationary spacetime:

In [17]:
```
K = gam.lie_derivative(b) / (2*N)
K.set_name('K')
print(K)
```

```
Field of symmetric bilinear forms K on the 3-dimensional differentiable
 manifold Sigma
```

The component $K_{13} = K_{x\phi}$:

In [18]:
```
K[1,3]
```

Out[18]:

$$2\Big(6\sqrt{3}\sqrt{2}x^{16} - 13824\left(\sqrt{3}\sqrt{2}x^2 + 10\sqrt{3}\sqrt{2}x + \sqrt{3}\sqrt{2}\right)y^{16} + 240\sqrt{3}\sqrt{2}x^{15}$$

$$+ 3793\sqrt{3}\sqrt{2}x^{14} - 6912$$
$$\left(\sqrt{3}\sqrt{2}x^4 + 20\sqrt{3}\sqrt{2}x^3 + 150\sqrt{3}\sqrt{2}x^2 + 500\sqrt{3}\sqrt{2}x + 817\sqrt{3}\sqrt{2}\right)y^{14}$$
$$+ 27650\sqrt{3}\sqrt{2}x^{13} + 72403\sqrt{3}\sqrt{2}x^{12} + 576$$
$$\left(27\sqrt{3}\sqrt{2}x^6 + 310\sqrt{3}\sqrt{2}x^5 + 1033\sqrt{3}\sqrt{2}x^4 + 1060\sqrt{3}\sqrt{2}x^3 + 10493\sqrt{3}\sqrt{2}x^2\right.$$
$$\left.+ 44870\sqrt{3}\sqrt{2}x + 69503\sqrt{3}\sqrt{2}\right)$$
$$- 81820\sqrt{3}\sqrt{2}x^{11} - 374975\sqrt{3}\sqrt{2}x^{10} - 96$$
$$\left(109\sqrt{3}\sqrt{2}x^8 + 520\sqrt{3}\sqrt{2}x^7 + 1504\sqrt{3}\sqrt{2}x^6 + 19360\sqrt{3}\sqrt{2}x^5 + 92770\right.\quad y^{1}$$
$$\sqrt{3}\sqrt{2}x^4 + 157960\sqrt{3}\sqrt{2}x^3 + 148264\sqrt{3}\sqrt{2}x^2 + 731920\sqrt{3}\sqrt{2}x + 1256425$$
$$\left.\sqrt{3}\sqrt{2}\right)$$
$$- 313810\sqrt{3}\sqrt{2}x^9 + 669975\sqrt{3}\sqrt{2}x^8 + 24$$
$$\left(9\sqrt{3}\sqrt{2}x^{10} + 250\sqrt{3}\sqrt{2}x^9 + 6873\sqrt{3}\sqrt{2}x^8 + 40920\sqrt{3}\sqrt{2}x^7 + 63402\sqrt{3}\sqrt{2}x\right.$$
$$+ 146220\sqrt{3}\sqrt{2}x^5 + 1047426\sqrt{3}\sqrt{2}x^4 + 2249400\sqrt{3}\sqrt{2}x^3 + 876525\sqrt{3}\sqrt{2}x^2$$
$$\left.+ 4308810\sqrt{3}\sqrt{2}x + 8401925\sqrt{3}\sqrt{2}\right)$$
$$+ 1617000\sqrt{3}\sqrt{2}x^7 + 999675\sqrt{3}\sqrt{2}x^6 + 96$$
$$\left(20\sqrt{3}\sqrt{2}x^{11} - 179\sqrt{3}\sqrt{2}x^{10} - 50\sqrt{3}\sqrt{2}x^9 - 2897\sqrt{3}\sqrt{2}x^8 - 28400\sqrt{3}\sqrt{2}x^7\right.$$
$$- 57446\sqrt{3}\sqrt{2}x^6 - 9020\sqrt{3}\sqrt{2}x^5 - 237650\sqrt{3}\sqrt{2}x^4 - 731060\sqrt{3}\sqrt{2}x^3$$
$$\left.- 267175\sqrt{3}\sqrt{2}x^2 - 1037250\sqrt{3}\sqrt{2}x - 2111325\sqrt{3}\sqrt{2}\right)$$
$$- 2277250\sqrt{3}\sqrt{2}x^5 - 4979375\sqrt{3}\sqrt{2}x^4$$
$$- \left(187\sqrt{3}\sqrt{2}x^{14} + 3590\sqrt{3}\sqrt{2}x^{13} - 5207\sqrt{3}\sqrt{2}x^{12} - 73540\sqrt{3}\sqrt{2}x^{11} - 4546\right.$$
$$\sqrt{3}\sqrt{2}x^{10} - 1150150\sqrt{3}\sqrt{2}x^9 + 199401\sqrt{3}\sqrt{2}x^8 - 1059000\sqrt{3}\sqrt{2}x^7$$
$$- 7811175\sqrt{3}\sqrt{2}x^6 + 2899610\sqrt{3}\sqrt{2}x^5 + 1675075\sqrt{3}\sqrt{2}x^4 - 32834500$$
$$\left.\sqrt{3}\sqrt{2}x^3 - 24681575\sqrt{3}\sqrt{2}x^2 - 69684250\sqrt{3}\sqrt{2}x - 122823125\sqrt{3}\sqrt{2}\right)$$
$$- 4037500\sqrt{3}\sqrt{2}x^3 + 3461875\sqrt{3}\sqrt{2}x^2 - 6$$
$$\left(\sqrt{3}\sqrt{2}x^{16} + 40\sqrt{3}\sqrt{2}x^{15} + 601\sqrt{3}\sqrt{2}x^{14} + 4010\sqrt{3}\sqrt{2}x^{13} + 12935\sqrt{3}\sqrt{2}x^{12}\right.$$
$$- 1060\sqrt{3}\sqrt{2}x^{11} + 10449\sqrt{3}\sqrt{2}x^{10} + 139590\sqrt{3}\sqrt{2}x^9 + 57825\sqrt{3}\sqrt{2}x^8$$
$$+ 146960\sqrt{3}\sqrt{2}x^7 + 781475\sqrt{3}\sqrt{2}x^6 - 702250\sqrt{3}\sqrt{2}x^5 - 2108075\sqrt{3}\sqrt{2}x^4$$
$$\left.- 348500\sqrt{3}\sqrt{2}x^3 + 2381875\sqrt{3}\sqrt{2}x^2 + 5456250\sqrt{3}\sqrt{2}x + 6941250\sqrt{3}\sqrt{2}\right)$$

$$+ 7231250\sqrt{3}\sqrt{2}x + 6109375\sqrt{3}\sqrt{2}\Big)$$

The type-(1,1) tensor $K^\sharp$ of components $K^i_{\ j} = \gamma^{ik} K_{kj}$:

In [19]:
```
Ku = K.up(gam, 0)
print(Ku)
```

```
Tensor field of type (1,1) on the 3-dimensional differentiable manifold
 Sigma
```

We may check that the hypersurface Σ is maximal, i.e. that $K^k_{\ k} = 0$:

In [20]:
```
trK = Ku.trace()
trK
```

Out[20]: 0

Connection and curvature

Let us call D the Levi-Civita connection associated with γ:

In [21]:
```
D = gam.connection(name='D')
print(D)
```

```
Levi-Civita connection D associated with the Riemannian metric gam on t
he 3-dimensional differentiable manifold Sigma
```

The Ricci tensor associated with γ:

In [22]:
```
Ric = gam.ricci()
print(Ric)
```

```
Field of symmetric bilinear forms Ric(gam) on the 3-dimensional differe
ntiable manifold Sigma
```

The scalar curvature $R = \gamma^{ij} R_{ij}$:

In [23]:
```
R = gam.ricci_scalar(name='R')
print(R)
```

```
Scalar field R on the 3-dimensional differentiable manifold Sigma
```

The coordinate expression of the Ricci scalar is huge:

In [24]:
```
R.expr()
```

Out[24]: 480000

$\Big(36\,x^{38} + 2880\,x^{37} - 191102976\,\left(3\,x^{4} + 20\,x^{3} - 6\,x^{2} - 60\,x - 101\right)y^{36}$

$+ 103116\,x^{36} + 2152440\,x^{35} + 382205952$
$\left(5\,x^{6} + 45\,x^{5} + 195\,x^{4} + 1210\,x^{3} + 3555\,x^{2} + 4065\,x - 291\right)y^{34} + 28527685$
$x^{34} + 243524500\,x^{33} - 63700992$
$\left(45\,x^{8} + 605\,x^{7} + 4434\,x^{6} + 21165\,x^{5} + 36233\,x^{4} - 28965\,x^{3} - 244428\,x^{2}\right.y^{32}$
$\left. - 457885\,x - 503220\right)$
$+ 1269998358\,x^{32} + 3199445660\,x^{31} + 10616832$
$\left(279\,x^{10} + 4375\,x^{9} + 30087\,x^{8} + 71400\,x^{7} - 238782\,x^{6} - 1698210\,x^{5}\right. \quad y^{30}$
$\left. - 6419238\,x^{4} - 18093840\,x^{3} - 36663561\,x^{2} - 47421645\,x - 39429937\right)$
$- 2269601041\,x^{30} - 34623715080\,x^{29} - 331776$
$\left(5729\,x^{12} + 49180\,x^{11} - 74502\,x^{10} - 3734980\,x^{9} - 26371593\,x^{8} - 109948680\right.y^{28}$
$x^{7} - 472804404\,x^{6} - 1606840680\,x^{5} - 3848737185\,x^{4} - 6740837780\,x^{3}$
$\left. - 9099798310\,x^{2} - 9444811860\,x - 7651340375\right)$
$- 59087224000\,x^{28} + 65688034640\,x^{27} - 110592$
$\left(2649\,x^{14} + 238580\,x^{13} + 3051495\,x^{12} + 19491840\,x^{11} + 91830265\,x^{10}\right. \quad y^{26}$
$+ 456692580\,x^{9} + 2183925951\,x^{8} + 7465185120\,x^{7} + 19582237971\,x^{6}$
$+ 43189401660\,x^{5} + 79002052285\,x^{4} + 114585899040\,x^{3} + 122428856475\,x^{2}$
$\left. + 101684757740\,x + 84896596125\right)$
$+ 384825320925\,x^{26} + 395426661500\,x^{25} + 9216$
$\left(181479\,x^{16} + 4048360\,x^{15} + 34156308\,x^{14} + 188537160\,x^{13} + 1036205812\right. \quad y^{24}$

3+1 Einstein equations

Let us check that the vacuum 3+1 Einstein equations are satisfied.

We start by the constraint equations:

Hamiltonian constraint

Let us first evaluate the term $K_{ij}K^{ij}$:

In [25]:
```
Kuu = Ku.up(gam, 1)
trKK = K['_ij']*Kuu['^ij']
print(trKK)
```

```
Scalar field on the 3-dimensional differentiable manifold Sigma
```

The vacuum Hamiltonian constraint equation is

$$R + K^2 - K_{ij}K^{ij} = 0$$

In [26]:
```
Ham = R + trK^2 - trKK
print(Ham)
Ham.display()
```

```
Scalar field zero on the 3-dimensional differentiable manifold Sigma
```

Out[26]:

$$\begin{array}{llcl} 0: & \Sigma & \longrightarrow & \mathbb{R} \\ & (x, y, \phi) & \longmapsto & 0 \end{array}$$

Hence the Hamiltonian constraint is satisfied.

Momentum constraint

In vaccum, the momentum constraint is

$$D_j K^j_{\ \ i} - D_i K = 0$$

In [27]:
```
mom = D(Ku).trace(0,2) - D(trK)
print(mom)
mom.display()
```

```
1-form on the 3-dimensional differentiable manifold Sigma
```

Out[27]: 0

Hence the momentum constraint is satisfied.

Dynamical Einstein equations

Let us first evaluate the symmetric bilinear form $k_{ij} := K_{ik}K^k_{\ \ j}$:

In [28]:
```
KK = K['_ik']*Ku['^k_j']
print(KK)
```

```
Tensor field of type (0,2) on the 3-dimensional differentiable manifold
 Sigma
```

In [29]:
```
KK1 = KK.symmetrize()
KK == KK1
```

Out[29]: True

In [30]:
```
KK = KK1
print(KK)
```

```
Field of symmetric bilinear forms on the 3-dimensional differentiable m
anifold Sigma
```

In vacuum and for stationary spacetimes, the dynamical Einstein equations are

$$\mathcal{L}_\beta K_{ij} - D_i D_j N + N \left(R_{ij} + K K_{ij} - 2 K_{ik} K^k_{\ \, j} \right) = 0$$

In [31]:
```
dyn = K.lie_derivative(b) - D(D(N)) + N*( Ric + trK*K - 2*KK )
print(dyn)
dyn.display()
```

```
Tensor field of type (0,2) on the 3-dimensional differentiable manifold
 Sigma
```

Out[31]: 0

Hence the dynamical Einstein equations are satisfied.

Finally we have checked that all the 3+1 Einstein equations are satisfied by the $\delta = 2$ Tomimatsu-Sato solution.