Reference manual of kerrgeodesic_gw

Release 0.4.2

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CONTENTS

CHAPTER

ONE

KERR SPACETIME

The Kerr black hole is implemented via the class *KerrBH*, which provides the Lorentzian manifold functionalities associated with the Kerr metric, as well as functionalities regarding circular orbits (e.g. computation of the ISCO, Roche limit, etc.).

REFERENCES:

- J. M. Bardeen, W. H. Press and S. A. Teukolsky, Astrophys. J. **178**, 347 (1972), doi:10.1086/151796
- L. Dai and R. Blandford, Mon. Not. Roy. Astron. Soc. **434**, 2948 (2013), doi:10.1093/mnras/stt1209

class kerrgeodesic_gw.kerr_spacetime.**KerrBH**(*a*, *m=1*, *manifold_name='M'*, *manifold_latex_name=None*, *metric_name='g'*, *metric_latex_name=None*)

Bases: `sage.manifolds.differentiable.pseudo_riemannian.PseudoRiemannianManifold`

Kerr black hole spacetime.

The Kerr spacetime is generated as a 4-dimensional Lorentzian manifold, endowed with the Boyer-Lindquist coordinates (default chart). Accordingly the class `KerrBH` inherits from the generic SageMath class PseudoRiemannianManifold.

INPUT:

- `a` – reduced angular momentum
- `m` – (default: `1`) total mass
- `manifold_name` – (default: `'M'`) string; name (symbol) given to the spacetime manifold
- `manifold_latex_name` – (default: `None`) string; LaTeX symbol to denote the spacetime manifold; if none is provided, it is set to `manifold_name`
- `metric_name` – (default: `'g'`) string; name (symbol) given to the metric tensor
- `metric_latex_name` – (default: `None`) string; LaTeX symbol to denote the metric tensor; if none is provided, it is set to `metric_name`

EXAMPLES:

Creating a Kerr spacetime with symbolic parameters (a, m):

```
sage: from kerrgeodesic_gw import KerrBH
sage: a, m = var('a m')
sage: BH = KerrBH(a, m); BH
Kerr spacetime M
sage: dim(BH)
4
```

The Boyer-Lindquist chart:

```
sage: BH.boyer_lindquist_coordinates()
Chart (M, (t, r, th, ph))
sage: latex(_)
\left(M,(t, r, {\theta}, {\phi})\right)
```

The Kerr metric:

```
sage: g = BH.metric(); g
Lorentzian metric g on the Kerr spacetime M
sage: g.display()
g = -(a^2*cos(th)^2 - 2*m*r + r^2)/(a^2*cos(th)^2 + r^2) dt⊗dt
 - 2*a*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2) dt⊗dph
 + (a^2*cos(th)^2 + r^2)/(a^2 - 2*m*r + r^2) dr⊗dr
 + (a^2*cos(th)^2 + r^2) dth⊗dth
 - 2*a*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2) dph⊗dt
 + (2*a^2*m*r*sin(th)^4 + (a^2*r^2 + r^4 + (a^4 + a^2*r^2)*cos(th)^2)*sin(th)^2)/(a^2*cos(th)^2 + r^2)␣
↪dph⊗dph
sage: g[0,3]
-2*a*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2)
```

A Kerr spacetime with specific numerical values for (a, m), namely $m = 1$ and $a = 0.9$:

```
sage: BH = KerrBH(0.9); BH
Kerr spacetime M
sage: g = BH.metric()
sage: g.display()  # tol 1.0e-13
g = -(r^2 + 0.81*cos(th)^2 - 2*r)/(r^2 + 0.81*cos(th)^2) dt⊗dt
 - 1.8*r*sin(th)^2/(r^2 + 0.81*cos(th)^2) dt⊗dph
 + (1.0*r^2 + 0.81*cos(th)^2)/(1.0*r^2 - 2.0*r + 0.81) dr⊗dr
 + (r^2 + 0.81*cos(th)^2) dth⊗dth
 - 1.8*r*sin(th)^2/(r^2 + 0.81*cos(th)^2) dph⊗dt
 + (1.62*r*sin(th)^4 + (1.0*r^4 + (0.81*r^2 + 0.6561)*cos(th)^2
    + 0.81*r^2)*sin(th)^2)/(1.0*r^2 + 0.81*cos(th)^2) dph⊗dph
sage: g[0,3]
-1.8*r*sin(th)^2/(r^2 + 0.81*cos(th)^2)
```

The Schwarrzschild spacetime as the special case $a = 0$ of Kerr spacetime:

```
sage: BH = KerrBH(0, m)
sage: g = BH.metric()
sage: g.display()
g = (2*m - r)/r dt⊗dt - r/(2*m - r) dr⊗dr + r^2 dth⊗dth + r^2*sin(th)^2 dph⊗dph
```

The object returned by *metric()* belongs to the SageMath class PseudoRiemannianMetric, for which many methods are available, like `christoffel_symbols_display()` to get the Christoffel symbols with respect to the Boyer-Lindquist coordinates (by default, only nonzero and non-redundant symbols are displayed):

```
sage: g.christoffel_symbols_display()
Gam^t_t,r = -m/(2*m*r - r^2)
Gam^r_t,t = -(2*m^2 - m*r)/r^3
Gam^r_r,r = m/(2*m*r - r^2)
Gam^r_th,th = 2*m - r
Gam^r_ph,ph = (2*m - r)*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
```

or `ricci()` to compute the Ricci tensor (identically zero here since we are dealing with a solution of Einstein equation in vacuum):

```
sage: g.ricci()
Field of symmetric bilinear forms Ric(g) on the Schwarzschild
```

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```
 spacetime M
sage: g.ricci().display()
Ric(g) = 0
```

Various methods of KerrBH class implement the computations of remarkable radii in the Kerr spacetime. Let us use them to reproduce Fig. 1 of the seminal article by Bardeen, Press and Teukolsky, ApJ **178**, 347 (1972), doi:10.1086/151796, which displays these radii as functions of the black hole spin paramater a. The radii of event horizon and inner (Cauchy) horizon are obtained by the methods *event_horizon_radius()* and *cauchy_horizon_radius()* respectively:

```
sage: graph = plot(lambda a: KerrBH(a).event_horizon_radius(),
....:              (0., 1.), color='black', thickness=1.5,
....:              legend_label=r"$r_{\rm H}$ (event horizon)",
....:              axes_labels=[r"$a/M$", r"$r/M$"],
....:              gridlines=True, frame=True, axes=False)
sage: graph += plot(lambda a: KerrBH(a).cauchy_horizon_radius(),
....:               (0., 1.), color='black', linestyle=':', thickness=1.5,
....:               legend_label=r"$r_{\rm C}$ (Cauchy horizon)")
```

The ISCO radius is computed by the method *isco_radius()*:

```
sage: graph += plot(lambda a: KerrBH(a).isco_radius(),
....:               (0., 1.), thickness=1.5,
....:               legend_label=r"$r_{\rm ISCO}$ (prograde)")
sage: graph += plot(lambda a: KerrBH(a).isco_radius(retrograde=True),
....:               (0., 1.), linestyle='--', thickness=1.5,
....:               legend_label=r"$r_{\rm ISCO}$ (retrograde)")
```

The radius of the marginally bound circular orbit is computed by the method *marginally_bound_orbit_radius()*:

```
sage: graph += plot(lambda a: KerrBH(a).marginally_bound_orbit_radius(),
....:               (0., 1.), color='purple', linestyle='-', thickness=1.5,
....:               legend_label=r"$r_{\rm mb}$ (prograde)")
sage: graph += plot(lambda a: KerrBH(a).marginally_bound_orbit_radius(retrograde=True),
....:               (0., 1.), color='purple', linestyle='--', thickness=1.5,
....:               legend_label=r"$r_{\rm mb}$ (retrograde)")
```

The radius of the photon circular orbit is computed by the method *photon_orbit_radius()*:

```
sage: graph += plot(lambda a: KerrBH(a).photon_orbit_radius(),
....:               (0., 1.), color='gold', linestyle='-', thickness=1.5,
....:               legend_label=r"$r_{\rm ph}$ (prograde)")
sage: graph += plot(lambda a: KerrBH(a).photon_orbit_radius(retrograde=True),
....:               (0., 1.), color='gold', linestyle='--', thickness=1.5,
....:               legend_label=r"$r_{\rm ph}$ (retrograde)")
```

The final plot:

```
sage: graph
Graphics object consisting of 8 graphics primitives
```

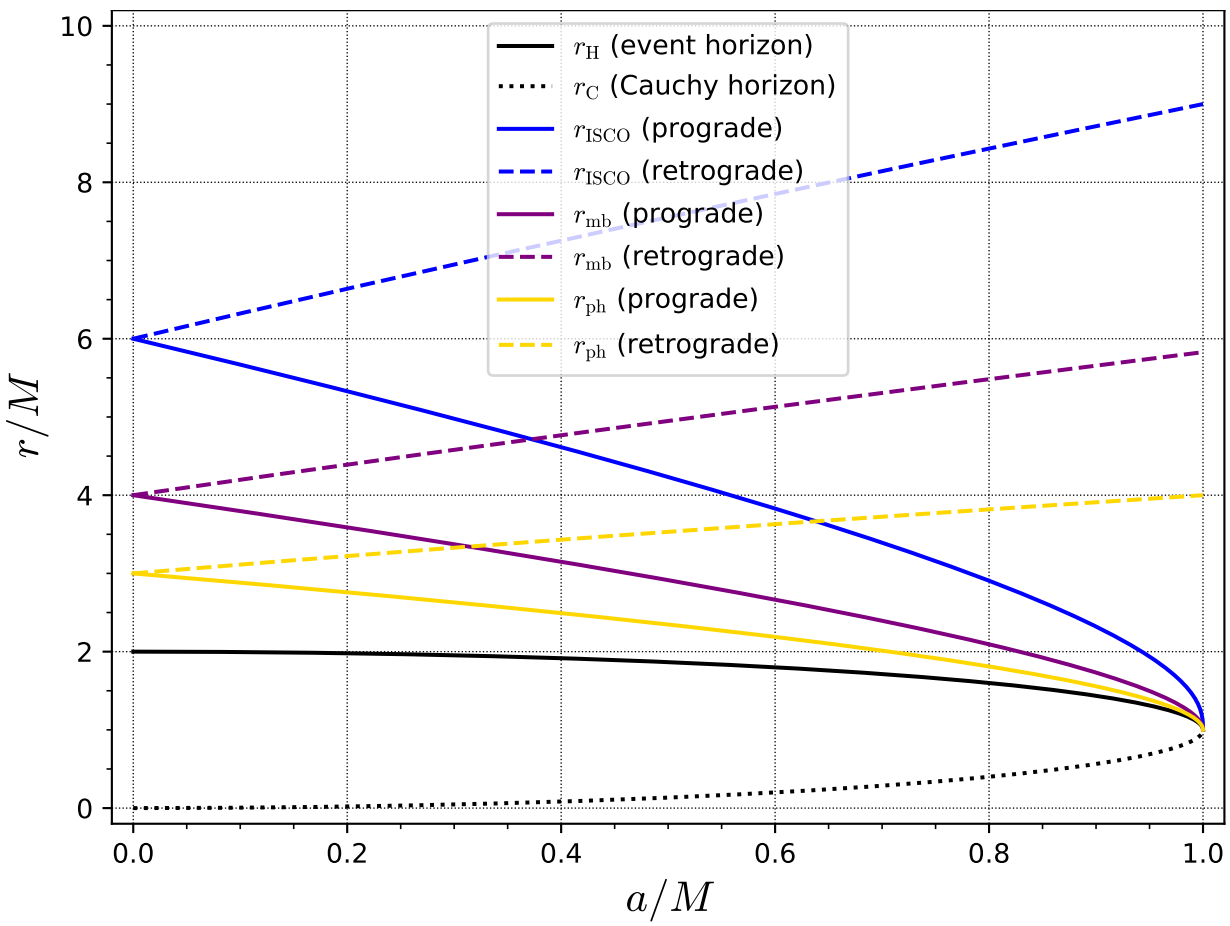
Arbitrary precision computations are possible:

```
sage: a = RealField(200)(0.95) # 0.95 with 200 bits of precision
sage: KerrBH(a).isco_radius()  # tol 1e-50
1.9372378781396625744170794927972658947432427390836799716847
```

angular_momentum()

Return the reduced angular momentum of the black hole.

EXAMPLES:


r_{H} (event horizon)
r_{C} (Cauchy horizon)
r_{ISCO} (prograde)
r_{ISCO} (retrograde)
r_{mb} (prograde)
r_{mb} (retrograde)
r_{ph} (prograde)
r_{ph} (retrograde)
r/M
a/M
0
2
4
6
8
10
0.0
0.2
0.4
0.6
0.8
1.0

```
sage: from kerrgeodesic_gw import KerrBH
sage: a, m = var('a m')
sage: BH = KerrBH(a, m)
sage: BH.spin()
a
```

An alias is `angular_momentum()`:

```
sage: BH.angular_momentum()
a
```

boyer_lindquist_coordinates(*symbols=None*, *names=None*)

Return the chart of Boyer-Lindquist coordinates.

INPUT:

- `symbols` – (default: `None`) string defining the coordinate text symbols and LaTeX symbols, with the same conventions as the argument `coordinates` in RealDiffChart; this is used only if the Boyer-Lindquist chart has not been already defined; if `None` the symbols are generated as (t, r, θ, ϕ).
- `names` – (default: `None`) unused argument, except if `symbols` is not provided; it must be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)

OUTPUT:

- the chart of Boyer-Lindquist coordinates, as an instance of RealDiffChart

EXAMPLES:

```
sage: from kerrgeodesic_gw import KerrBH
sage: a, m = var('a m')
sage: BH = KerrBH(a, m)
sage: BH.boyer_lindquist_coordinates()
Chart (M, (t, r, th, ph))
sage: latex(BH.boyer_lindquist_coordinates())
\left(M,(t, r, {\theta}, {\phi})\right)
```

The coordinate variables are returned by the square bracket operator:

```
sage: BH.boyer_lindquist_coordinates()[0]
t
sage: BH.boyer_lindquist_coordinates()[1]
r
sage: BH.boyer_lindquist_coordinates()[:]
(t, r, th, ph)
```

They can also be obtained via the operator `<,>` at the same time as the chart itself:

```
sage: BLchart.<t, r, th, ph> = BH.boyer_lindquist_coordinates()
sage: BLchart
Chart (M, (t, r, th, ph))
sage: type(ph)
<type 'sage.symbolic.expression.Expression'>
```

Actually, `BLchart.<t, r, th, ph> = BH.boyer_lindquist_coordinates()` is a shortcut for:

```
sage: BLchart = BH.boyer_lindquist_coordinates()
sage: t, r, th, ph = BLchart[:]
```

The coordinate symbols can be customized:

```
sage: BH = KerrBH(a)
sage: BH.boyer_lindquist_coordinates(symbols=r"T R Th:\Theta Ph:\Phi")
Chart (M, (T, R, Th, Ph))
sage: latex(BH.boyer_lindquist_coordinates())
\left(M,(T, R, {\Theta}, {\Phi})\right)
```

cauchy_horizon_radius()

Return the value of the Boyer-Lindquist coordinate r at the inner horizon (Cauchy horizon).

EXAMPLES:

```
sage: from kerrgeodesic_gw import KerrBH
sage: a, m = var('a m')
sage: BH = KerrBH(a, m)
sage: BH.inner_horizon_radius()
m - sqrt(-a^2 + m^2)
```

An alias is `cauchy_horizon_radius()`:

```
sage: BH.cauchy_horizon_radius()
m - sqrt(-a^2 + m^2)
```

event_horizon_radius()

Return the value of the Boyer-Lindquist coordinate r at the black hole event horizon.

EXAMPLES:

```
sage: from kerrgeodesic_gw import KerrBH
sage: a, m = var('a m')
sage: BH = KerrBH(a, m)
sage: BH.outer_horizon_radius()
m + sqrt(-a^2 + m^2)
```

An alias is `event_horizon_radius()`:

```
sage: BH.event_horizon_radius()
m + sqrt(-a^2 + m^2)
```

The horizon radius of the Schwarzschild black hole:

```
sage: assume(m>0)
sage: KerrBH(0, m).event_horizon_radius()
2*m
```

The horizon radius of the extreme Kerr black hole ($a = m$):

```
sage: KerrBH(m, m).event_horizon_radius()
m
```

geodesic(*parameter_range*, *initial_point*, *pt0=None*, *pr0=None*, *pth0=None*, *pph0=None*, *mu=None*, *E=None*, *L=None*, *Q=None*, *r_increase=True*, *th_increase=True*, *chart=None*, *name=None*, *latex_name=None*, *a_num=None*, *m_num=None*, *verbose=False*)

Construct a geodesic on `self`.

INPUT:

- `parameter_range` – range of the affine parameter λ, as a pair `(lambda_min, lambda_max)`
- `initial_point` – point of Kerr spacetime from which the geodesic is to be integrated
- `pt0` – (default: `None`) Boyer-Lindquist component p^t of the initial 4-momentum vector
- `pr0` – (default: `None`) Boyer-Lindquist component p^r of the initial 4-momentum vector

- `pth0` – (default: `None`) Boyer-Lindquist component p^θ of the initial 4-momentum vector
- `pph0` – (default: `None`) Boyer-Lindquist component p^ϕ of the initial 4-momentum vector
- `mu` – (default: `None`) mass μ of the particle
- `E` – (default: `None`) conserved energy E of the particle
- `L` – (default: `None`) conserved angular momemtum L of the particle
- `Q` – (default: `None`) Carter constant Q of the particle
- `r_increase` – (default: `True`) boolean; if `True`, the initial value of $p^r = \mathrm{d}r/\mathrm{d}\lambda$ determined from the integral of motions is positive or zero, otherwise, p^r is negative
- `th_increase` – (default: `True`) boolean; if `True`, the initial value of $p^\theta = \mathrm{d}\theta/\mathrm{d}\lambda$ determined from the integral of motions is positive or zero, otherwise, p^θ is negative
- `chart` – (default: `None`) chart on the spacetime manifold in terms of which the geodesic equations are expressed; if `None` the default chart (Boyer-Lindquist coordinates) is assumed
- `name` – (default: `None`) string; symbol given to the geodesic
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the geodesic; if none is provided, `name` will be used
- `a_num` – (default: `None`) numerical value of the Kerr spin parameter a (required for a numerical integration)
- `m_num` – (default: `None`) numerical value of the Kerr mass parameter m (required for a numerical integration)
- `verbose` – (default: `False`) boolean; determines whether some information is printed during the construction of the geodesic

OUTPUT:

- an instance of *`KerrGeodesic`*

EXAMPLES:

See *`KerrGeodesic`*.

`inner_horizon_radius()`

Return the value of the Boyer-Lindquist coordinate r at the inner horizon (Cauchy horizon).

EXAMPLES:

```
sage: from kerrgeodesic_gw import KerrBH
sage: a, m = var('a m')
sage: BH = KerrBH(a, m)
sage: BH.inner_horizon_radius()
m - sqrt(-a^2 + m^2)
```

An alias is `cauchy_horizon_radius()`:

```
sage: BH.cauchy_horizon_radius()
m - sqrt(-a^2 + m^2)
```

`isco_radius`(*retrograde=False*)

Return the Boyer-Lindquist radial coordinate of the innermost stable circular orbit (ISCO) in the equatorial plane.

INPUT:

- `retrograde` – (default: `False`) boolean determining whether retrograde or prograde (direct) orbits are considered

OUTPUT:

- Boyer-Lindquist radial coordinate r of the innermost stable circular orbit in the equatorial plane

EXAMPLES:

```
sage: from kerrgeodesic_gw import KerrBH
sage: a, m = var('a m')
sage: BH = KerrBH(a, m)
sage: BH.isco_radius()
m*(sqrt((((a/m + 1)^(1/3) + (-a/m + 1)^(1/3))*(-a^2/m^2 + 1)^(1/3) + 1)^2
 + 3*a^2/m^2) - sqrt(-(((a/m + 1)^(1/3) + (-a/m + 1)^(1/3))*(-a^2/m^2 + 1)^(1/3)
 + 2*sqrt((((a/m + 1)^(1/3) + (-a/m + 1)^(1/3))*(-a^2/m^2 + 1)^(1/3) + 1)^2
 + 3*a^2/m^2) + 4)*(((a/m + 1)^(1/3) + (-a/m + 1)^(1/3))*(-a^2/m^2 + 1)^(1/3) - 2)) + 3)
sage: BH.isco_radius(retrograde=True)
m*(sqrt((((a/m + 1)^(1/3) + (-a/m + 1)^(1/3))*(-a^2/m^2 + 1)^(1/3) + 1)^2
 + 3*a^2/m^2) + sqrt(-(((a/m + 1)^(1/3) + (-a/m + 1)^(1/3))*(-a^2/m^2 + 1)^(1/3)
 + 2*sqrt((((a/m + 1)^(1/3) + (-a/m + 1)^(1/3))*(-a^2/m^2 + 1)^(1/3) + 1)^2
 + 3*a^2/m^2) + 4)*(((a/m + 1)^(1/3) + (-a/m + 1)^(1/3))*(-a^2/m^2 + 1)^(1/3) - 2)) + 3)
sage: KerrBH(0.5).isco_radius()  # tol 1.0e-13
4.23300252953083
sage: KerrBH(0.9).isco_radius()  # tol 1.0e-13
2.32088304176189
sage: KerrBH(0.98).isco_radius()  # tol 1.0e-13
1.61402966763547
```

ISCO in Schwarzschild spacetime:

```
sage: KerrBH(0, m).isco_radius()
6*m
```

ISCO in extreme Kerr spacetime ($a = m$):

```
sage: KerrBH(m, m).isco_radius()
m
sage: KerrBH(m, m).isco_radius(retrograde=True)
9*m
```

map_to_Euclidean()

Map from Kerr spacetime to the Euclidean 4-space, based on Boyer-Lindquist coordinates

marginally_bound_orbit_radius(*retrograde=False*)

Return the Boyer-Lindquist radial coordinate of the marginally bound circular orbit in the equatorial plane.

INPUT:

- `retrograde` – (default: `False`) boolean determining whether retrograde or prograde (direct) orbits are considered

OUTPUT:

- Boyer-Lindquist radial coordinate r of the marginally bound circular orbit in the equatorial plane

EXAMPLES:

```
sage: from kerrgeodesic_gw import KerrBH
sage: a, m = var('a m')
sage: BH = KerrBH(a, m)
sage: BH.marginally_bound_orbit_radius()
-a + 2*sqrt(-a + m)*sqrt(m) + 2*m
sage: BH.marginally_bound_orbit_radius(retrograde=True)
a + 2*sqrt(a + m)*sqrt(m) + 2*m
```

Marginally bound orbit in Schwarzschild spacetime:

```
sage: KerrBH(0, m).marginally_bound_orbit_radius()
4*m
```

Marginally bound orbits in extreme Kerr spacetime ($a = m$):

```
sage: KerrBH(m, m).marginally_bound_orbit_radius()
m
sage: KerrBH(m, m).marginally_bound_orbit_radius(retrograde=True)
2*sqrt(2)*m + 3*m
```

mass()

Return the (ADM) mass of the black hole.

EXAMPLES:

```
sage: from kerrgeodesic_gw import KerrBH
sage: a, m = var('a m')
sage: BH = KerrBH(a, m)
sage: BH.mass()
m
sage: KerrBH(a).mass()
1
```

metric()

Return the metric tensor.

EXAMPLES:

```
sage: from kerrgeodesic_gw import KerrBH
sage: a, m = var('a m')
sage: BH = KerrBH(a, m)
sage: BH.metric()
Lorentzian metric g on the Kerr spacetime M
sage: BH.metric().display()
g = -(a^2*cos(th)^2 - 2*m*r + r^2)/(a^2*cos(th)^2 + r^2) dt⊗dt
 - 2*a*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2) dt⊗dph
 + (a^2*cos(th)^2 + r^2)/(a^2 - 2*m*r + r^2) dr⊗dr
 + (a^2*cos(th)^2 + r^2) dth⊗dth
 - 2*a*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2) dph⊗dt
 + (2*a^2*m*r*sin(th)^4 + (a^2*r^2 + r^4 + (a^4 + a^2*r^2)*cos(th)^2)*sin(th)^2)/(a^2*cos(th)^2 + r^
↪2) dph⊗dph
```

The Schwarzschild metric:

```
sage: KerrBH(0, m).metric().display()
g = (2*m - r)/r dt⊗dt - r/(2*m - r) dr⊗dr + r^2 dth⊗dth
 + r^2*sin(th)^2 dph⊗dph
```

orbital_angular_velocity(*r*, *retrograde=False*)

Return the angular velocity on a circular orbit.

The angular velocity Ω on a circular orbit of Boyer-Lindquist radial coordinate r around a Kerr black hole of parameters (m, a) is given by the formula

$$\Omega := \frac{\mathrm{d}\phi}{\mathrm{d}t} = \pm\frac{m^{1/2}}{r^{3/2} \pm am^{1/2}} \tag{1.1}$$

where (t, ϕ) are the Boyer-Lindquist time and azimuthal coordinates and $\pm$ is $+$ (resp. $-$) for a prograde (resp. retrograde) orbit.

INPUT:

- `r` – Boyer-Lindquist radial coordinate r of the circular orbit
- `retrograde` – (default: `False`) boolean determining whether the orbit is retrograde or prograde

OUTPUT:

- Angular velocity Ω computed according to Eq. (1.1)

EXAMPLES:

```
sage: from kerrgeodesic_gw import KerrBH
sage: a, m, r = var('a m r')
sage: BH = KerrBH(a, m)
sage: BH.orbital_angular_velocity(r)
sqrt(m)/(a*sqrt(m) + r^(3/2))
sage: BH.orbital_angular_velocity(r, retrograde=True)
sqrt(m)/(a*sqrt(m) - r^(3/2))
sage: KerrBH(0.9).orbital_angular_velocity(4.)  # tol 1.0e-13
0.112359550561798
```

Orbital angular velocity around a Schwarzschild black hole:

```
sage: KerrBH(0, m).orbital_angular_velocity(r)
sqrt(m)/r^(3/2)
```

Orbital angular velocity on the prograde ISCO of an extreme Kerr black hole ($a = m$):

```
sage: EKBH = KerrBH(m, m)
sage: EKBH.orbital_angular_velocity(EKBH.isco_radius())
1/2/m
```

orbital_frequency(*r*, *retrograde=False*)

Return the orbital frequency of a circular orbit.

The frequency f of a circular orbit of Boyer-Lindquist radial coordinate r around a Kerr black hole of parameters (m, a) is $f = \Omega/(2\pi)$, where Ω is given by Eq. (1.1).

INPUT:

- `r` – Boyer-Lindquist radial coordinate r of the circular orbit
- `retrograde` – (default: `False`) boolean determining whether the orbit is retrograde or prograde

OUTPUT:

- orbital frequency f

EXAMPLES:

```
sage: from kerrgeodesic_gw import KerrBH
sage: a, m, r = var('a m r')
sage: BH = KerrBH(a, m)
sage: BH.orbital_frequency(r)
1/2*sqrt(m)/(pi*(a*sqrt(m) + r^(3/2)))
sage: BH.orbital_frequency(r, retrograde=True)
1/2*sqrt(m)/(pi*(a*sqrt(m) - r^(3/2)))
sage: KerrBH(0.9).orbital_frequency(4.)  # tol 1.0e-13
0.0178825778754939
sage: KerrBH(0.9).orbital_frequency(float(4))  # tol 1.0e-13
0.0178825778754939
```

Orbital angular velocity around a Schwarzschild black hole:

```
sage: KerrBH(0, m).orbital_frequency(r)
1/2*sqrt(m)/(pi*r^(3/2))
```

Orbital angular velocity on the prograde ISCO of an extreme Kerr black hole ($a = m$):

```
sage: EKBH = KerrBH(m, m)
sage: EKBH.orbital_frequency(EKBH.isco_radius())
1/4/(pi*m)
```

For numerical values, the outcome depends on the type of the entry:

```
sage: KerrBH(0).orbital_frequency(6)
1/72*sqrt(6)/pi
sage: KerrBH(0).orbital_frequency(6.)  # tol 1.0e-13
0.0108291222393566
sage: KerrBH(0).orbital_frequency(RealField(200)(6))  # tol 1e-50
0.01082912223935661260980372292046189945754815231296101704318 0
sage: KerrBH(0.5).orbital_frequency(RealField(200)(6))  # tol 1.0e-13
0.0104728293495021
sage: KerrBH._clear_cache_()  # to remove the BH object created with a=0.5
sage: KerrBH(RealField(200)(0.5)).orbital_frequency(RealField(200)(6))  # tol 1.0e-50
0.01047282934950211196214675443399079073841562492110939261623 7
```

outer_horizon_radius()

Return the value of the Boyer-Lindquist coordinate r at the black hole event horizon.

EXAMPLES:

```
sage: from kerrgeodesic_gw import KerrBH
sage: a, m = var('a m')
sage: BH = KerrBH(a, m)
sage: BH.outer_horizon_radius()
m + sqrt(-a^2 + m^2)
```

An alias is `event_horizon_radius()`:

```
sage: BH.event_horizon_radius()
m + sqrt(-a^2 + m^2)
```

The horizon radius of the Schwarzschild black hole:

```
sage: assume(m>0)
sage: KerrBH(0, m).event_horizon_radius()
2*m
```

The horizon radius of the extreme Kerr black hole ($a = m$):

```
sage: KerrBH(m, m).event_horizon_radius()
m
```

photon_orbit_radius(*retrograde=False*)

Return the Boyer-Lindquist radial coordinate of the circular orbit of photons in the equatorial plane.

INPUT:

- `retrograde` – (default: `False`) boolean determining whether retrograde or prograde (direct) orbits are considered

OUTPUT:

- Boyer-Lindquist radial coordinate r of the circular orbit of photons in the equatorial plane

EXAMPLES:

```
sage: from kerrgeodesic_gw import KerrBH
sage: a, m = var('a m')
```

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```
sage: BH = KerrBH(a, m)
sage: BH.photon_orbit_radius()
2*m*(cos(2/3*arccos(-a/m)) + 1)
sage: BH.photon_orbit_radius(retrograde=True)
2*m*(cos(2/3*arccos(a/m)) + 1)
```

Photon orbit in Schwarzschild spacetime:

```
sage: KerrBH(0, m).photon_orbit_radius()
3*m
```

Photon orbits in extreme Kerr spacetime ($a = m$):

```
sage: KerrBH(m, m).photon_orbit_radius()
m
sage: KerrBH(m, m).photon_orbit_radius(retrograde=True)
4*m
```

roche_limit_radius(*rho*, *rho_unit='solar'*, *mass_bh=None*, *k_rot=0*, *r_min=None*, *r_max=50*)

Evaluate the orbital radius of the Roche limit for a star of given density and rotation state.

The *Roche limit* is defined as the orbit at which the star fills its Roche volume (cf. `roche_volume()`).

INPUT:

- `rho` – mean density of the star, in units of `rho_unit`
- `rho_unit` – (default: `'solar'`) string specifying the unit in which `rho` is provided; allowed values are
  - `'solar'`: density of the sun ($1.41 \times 10^3\ \mathrm{kg\,m^{-3}}$)
  - `'SI'`: SI unit ($\mathrm{kg\,m^{-3}}$)
  - `'M^{-2}'`: inverse square of the black hole mass M (geometrized units)
- `mass_bh` – (default: `None`) black hole mass M in solar masses; must be set if `rho_unit = 'solar'` or `'SI'`
- `k_rot` – (default: `0`) rotational parameter $k_\omega := \omega/\Omega$, where ω is the angular velocity of the star (assumed to be a rigid rotator) with respect to some inertial frame and Ω is the orbital angular velocity (cf. `roche_volume()`)
- `r_min` – (default: `None`) lower bound for the search of the Roche limit radius, in units of M; if none is provided, the value of r at the prograde ISCO is used
- `r_max` – (default: `50`) upper bound for the search of the Roche limit radius, in units of M

OUPUT:

- Boyer-Lindquist radial coordinate r of the circular orbit at which the star fills its Roche volume, in units of the black hole mass M

EXAMPLES:

Roche limit of a non-rotating solar type star around a Schwarzschild black hole of mass equal to that of Sgr A* ($4.1\ 10^6\ M_\odot$):

```
sage: from kerrgeodesic_gw import KerrBH
sage: BH = KerrBH(0)
sage: BH.roche_limit_radius(1, mass_bh=4.1e6)  # tol 1.0e-13
34.23653024850463
```

Instead of providing the density in solar units (the default), we can provide it in SI units ($\mathrm{kg\,m^{-3}}$):

```
sage: BH.roche_limit_radius(1.41e3, rho_unit='SI', mass_bh=4.1e6)  # tol 1.0e-13
34.23517310503541
```

or in geometrized units (M^{-2}), in which case it is not necessary to specify the black hole mass:

```
sage: BH.roche_limit_radius(3.84e-5, rho_unit='M^{-2}')  # tol 1.0e-13
34.22977166547967
```

Case of a corotating star:

```
sage: BH.roche_limit_radius(1, mass_bh=4.1e6, k_rot=1)  # tol 1.0e-13
37.72150497210367
```

Case of a brown dwarf:

```
sage: BH.roche_limit_radius(131., mass_bh=4.1e6)  # tol 1.0e-13
7.3103232747243165
sage: BH.roche_limit_radius(131., mass_bh=4.1e6, k_rot=1)  # tol 1.0e-13
7.858389409707688
```

Case of a white dwarf:

```
sage: BH.roche_limit_radius(1.1e6, mass_bh=4.1e6, r_min=0.1)  # tol 1.0e-13
0.2848049914201514
sage: BH.roche_limit_radius(1.1e6, mass_bh=4.1e6, k_rot=1, r_min=0.1)  # tol 1.0e-13
0.3264724605157346
```

Roche limits around a rapidly rotating black hole:

```
sage: BH = KerrBH(0.999)
sage: BH.roche_limit_radius(1, mass_bh=4.1e6)  # tol 1.0e-13
34.250609907563984
sage: BH.roche_limit_radius(1, mass_bh=4.1e6, k_rot=1)  # tol 1.0e-13
37.72350054417335
sage: BH.roche_limit_radius(64.2, mass_bh=4.1e6)  # tol 1.0e-13
8.74356702311824
sage: BH.roche_limit_radius(64.2, mass_bh=4.1e6, k_rot=1)  # tol 1.0e-13
9.575613156232857
```

roche_volume(*r0*, *k_rot*)

Roche volume of a star on a given circular orbit.

The Roche volume depends on the rotational parameter

$$k_\omega := \frac{\omega}{\Omega}$$

where ω is the angular velocity of the star (assumed to be a rigid rotator) with respect to some inertial frame and Ω is the orbital angular velocity (cf. `orbital_angular_velocity()`). The Roche volume is computed according to formulas based on the Kerr metric and established by Dai & Blandford, Mon. Not. Roy. Astron. Soc. **434**, 2948 (2013), doi:10.1093/mnras/stt1209.

INPUT:

- `r0` – Boyer-Lindquist radial coordinate r of the circular orbit, in units of the black hole mass
- `k_rot` – rotational parameter k_ω defined above

OUPUT:

- the dimensionless quantity $V_R/(\mu M^2)$, where V_R is the Roche volume, μ the mass of the star and M the mass of the central black hole

EXAMPLES:

Roche volume around a Schwarzschild black hole:

```
sage: from kerrgeodesic_gw import KerrBH
sage: BH = KerrBH(0)
sage: BH.roche_volume(6, 0)  # tol 1.0e-13
98.49600000000001
```

Comparison with Eq. (26) of Dai & Blandford (2013) doi:10.1093/mnras/stt1209:

```
sage: _ - 0.456*6^3  # tol 1.0e-13
0.000000000000000
```

Case $k_\omega = 1$:

```
sage: BH.roche_volume(6, 1)  # tol 1.0e-13
79.145115913556
```

Comparison with Eq. (26) of Dai & Blandford (2013):

```
sage: _ - 0.456/(1+1/4.09)*6^3  # tol 1.0e-13
0.000000000000000
```

Newtonian limit:

```
sage: BH.roche_volume(1000, 0)  # tol 1.0e-13
681633403.5759227
```

Comparison with Eq. (10) of Dai & Blandford (2013):

```
sage: _ - 0.683*1000^3  # tol 1.0e-13
-1.36659642407727e6
```

Roche volume around a rapidly rotating Kerr black hole:

```
sage: BH = KerrBH(0.9)
sage: rI = BH.isco_radius()
sage: BH.roche_volume(rI, 0)  # tol 1.0e-13
5.70065302837734
```

Comparison with Eq. (26) of Dai & Blandford (2013):

```
sage: _ - 0.456*rI^3  # tol 1.0e-13
0.000000000000000
```

Case $k_\omega = 1$:

```
sage: BH.roche_volume(rI, 1)  # tol 1.0e-13
4.58068190295940
```

Comparison with Eq. (26) of Dai & Blandford (2013):

```
sage: _ - 0.456/(1+1/4.09)*rI^3  # tol 1.0e-13
0.000000000000000
```

Newtonian limit:

```
sage: BH.roche_volume(1000, 0)  # tol 1.0e-13
6.81881451514361e8
```

Comparison with Eq. (10) of Dai & Blandford (2013):

```
sage: _ - 0.683*1000^3  # tol 1.0e-13
-1.11854848563898e6
```

spin()

Return the reduced angular momentum of the black hole.

EXAMPLES:

```
sage: from kerrgeodesic_gw import KerrBH
sage: a, m = var('a m')
sage: BH = KerrBH(a, m)
sage: BH.spin()
a
```

An alias is `angular_momentum()`:

```
sage: BH.angular_momentum()
a
```

CHAPTER

TWO

GEODESICS IN KERR SPACETIME

Geodesics of Kerr spacetime are implemented via the class `KerrGeodesic`.

class kerrgeodesic_gw.kerr_geodesic.**KerrGeodesic**(*parent*, *initial_point*, *pt0=None*, *pr0=None*, *pth0=None*, *pph0=None*, *mu=None*, *E=None*, *L=None*, *Q=None*, *r_increase=True*, *th_increase=True*, *chart=None*, *name=None*, *latex_name=None*, *a_num=None*, *m_num=None*, *verbose=False*)

Bases: `sage.manifolds.differentiable.integrated_curve.IntegratedGeodesic`

Geodesic of Kerr spacetime.

Geodesics are computed by solving the geodesic equation via the generic SageMath geodesic integrator.

INPUT:

- `parent` – IntegratedGeodesicSet, the set of curves $\mathrm{Hom}_{\mathrm{geodesic}}(I, M)$ to which the geodesic belongs
- `initial_point` – point of Kerr spacetime from which the geodesic is to be integrated
- `pt0` – (default: `None`) Boyer-Lindquist component p^t of the initial 4-momentum vector
- `pr0` – (default: `None`) Boyer-Lindquist component p^r of the initial 4-momentum vector
- `pth0` – (default: `None`) Boyer-Lindquist component p^θ of the initial 4-momentum vector
- `pph0` – (default: `None`) Boyer-Lindquist component p^ϕ of the initial 4-momentum vector
- `mu` – (default: `None`) mass μ of the particle
- `E` – (default: `None`) conserved energy E of the particle
- `L` – (default: `None`) conserved angular momemtum L of the particle
- `Q` – (default: `None`) Carter constant Q of the particle
- `r_increase` – (default: `True`) boolean; if `True`, the initial value of $p^r = \mathrm{d}r/\mathrm{d}\lambda$ determined from the integral of motions is positive or zero, otherwise, p^r is negative
- `th_increase` – (default: `True`) boolean; if `True`, the initial value of $p^\theta = \mathrm{d}\theta/\mathrm{d}\lambda$ determined from the integral of motions is positive or zero, otherwise, p^θ is negative
- `chart` – (default: `None`) chart on the spacetime manifold in terms of which the geodesic equations are expressed; if `None` the default chart (Boyer-Lindquist coordinates) is assumed
- `name` – (default: `None`) string; symbol given to the geodesic
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the geodesic; if none is provided, `name` will be used
- `a_num` – (default: `None`) numerical value of the Kerr spin parameter a (required for a numerical integration)

- m_num – (default: None) numerical value of the Kerr mass parameter m (required for a numerical integration)
- verbose – (default: False) boolean; determines whether some information is printed during the construction of the geodesic

EXAMPLES:

We construct first the Kerr spacetime:

```
sage: from kerrgeodesic_gw import KerrBH
sage: a = var('a')
sage: M = KerrBH(a); M
Kerr spacetime M
sage: BLchart = M.boyer_lindquist_coordinates(); BLchart
Chart (M, (t, r, th, ph))
```

We pick an initial spacetime point for the geodesic:

```
sage: init_point = M((0, 6, pi/2, 0), name='P')
```

A geodesic is constructed by providing the range of the affine parameter, the initial point and either (i) the Boyer-Lindquist components $(p_0^t, p_0^r, p_0^\theta, p_0^\phi)$ of the initial 4-momentum vector $p_0 = \frac{\mathrm{d}x}{\mathrm{d}\lambda}\big|_{\lambda=0}$, (ii) the four integral of motions (μ, E, L, Q) or (iii) some of the components of p_0 along with with some integrals of motion. We shall also specify some numerical value for the Kerr spin parameter a. Examples of (i) and (iii) are provided below. Here, we choose $\lambda \in [0, 300m]$, the option (ii) and $a = 0.998m$, where m in the black hole mass:

```
sage: geod = M.geodesic([0, 300], init_point, mu=1, E=0.883,
....:                   L=1.982, Q=0.467, a_num=0.998)
sage: geod
Geodesic of the Kerr spacetime M
```

The numerical integration of the geodesic equation is performed via *integrate()*, by providing the step in $\delta\lambda$ in units of m:

```
sage: geod.integrate(step=0.005)
```

We can then plot the geodesic:

```
sage: geod.plot()
Graphics3d Object
```

Actually, many options can be passed to *plot()*. For instance to a get a 3D spacetime diagram:

```
sage: geod.plot(coordinates='txy')
Graphics3d Object
```

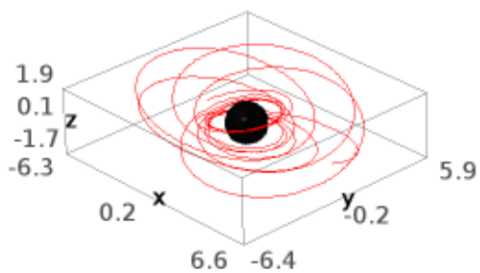
or to get the trace of the geodesic in the (x, y) plane:

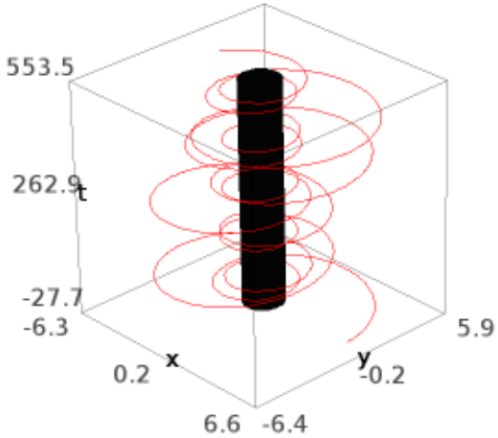
```
sage: geod.plot(coordinates='xy', plot_points=2000)
Graphics object consisting of 2 graphics primitives
```

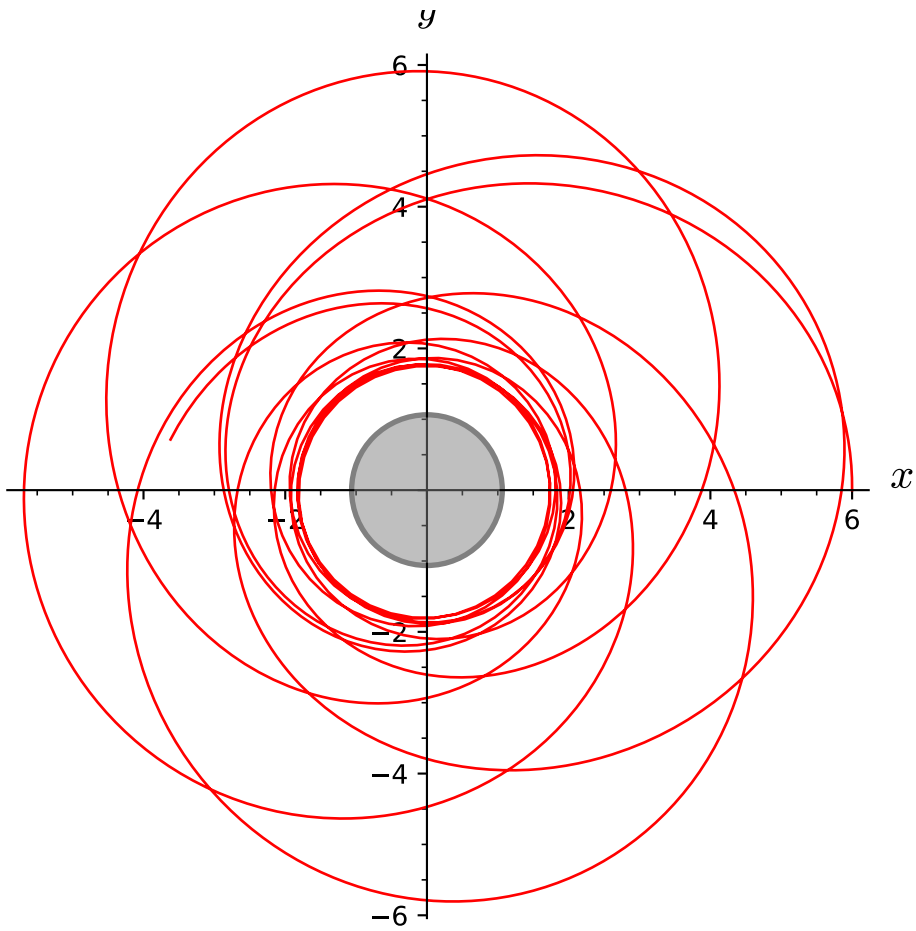
or else to get the trace in the (x, z) plane:

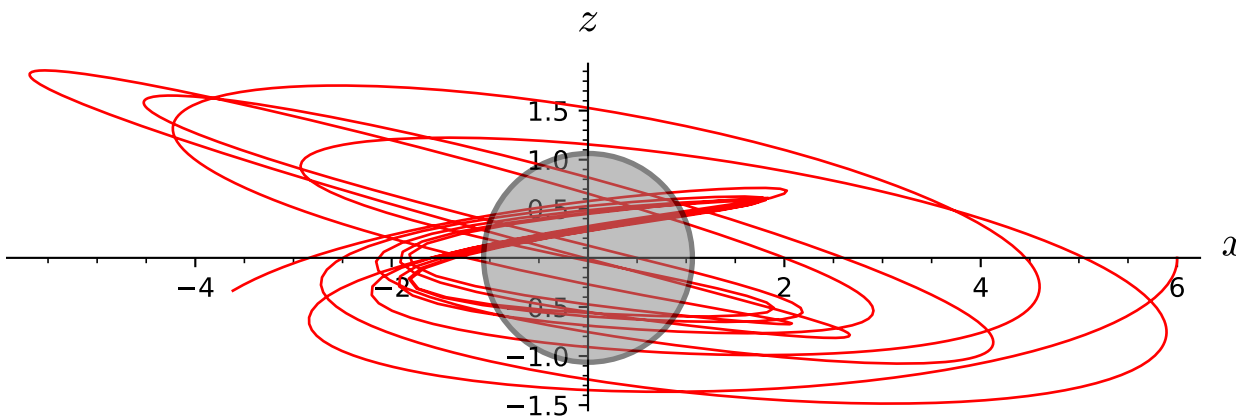
```
sage: geod.plot(coordinates='xz')
Graphics object consisting of 2 graphics primitives
```

As a curve, the geodesic is a map from an interval of $\mathbb{R}$ to the spacetime M:


1.9
0.1
z
-1.7
-6.3
0.2
x
6.6
-6.4
y
-0.2
5.9


553.5
262.9
t
-27.7
-6.3
0.2
x
6.6
-6.4
y
-0.2
5.9


y
6
4
2
−2
−4
−6
x
−4
−2
2
4
6


z
x
1.5
1.0
0.5
−0.5
−1.0
−1.5
−4
−2
2
4
6

```
sage: geod.display()
(0, 300) → M
sage: geod.domain()
Real interval (0, 300)
sage: geod.codomain()
Kerr spacetime M
```

It maps values of λ to spacetime points:

```
sage: geod(0)
Point on the Kerr spacetime M
sage: geod(0).coordinates()  # coordinates in the default chart  # tol 1.0e-13
(0.0, 6.0, 1.5707963267948966, 0.0)
sage: BLchart(geod(0))       # equivalent to above   # tol 1.0e-13
(0.0, 6.0, 1.5707963267948966, 0.0)
sage: geod(300).coordinates()   # tol 1.0e-13
(553.4637326813786, 3.703552505462962, 1.6613834863942039, 84.62814710987239)
```

The initial 4-momentum vector p_0 is returned by the method `initial_tangent_vector()`:

```
sage: p0 = geod.initial_tangent_vector(); p0
Tangent vector p at Point P on the Kerr spacetime M
sage: p0 in M.tangent_space(init_point)
True
sage: p0.display()  # tol 1.0e-13
p = 1.29225788954106 ∂/∂t + 0.00438084990626460 ∂/∂r
 + 0.0189826106258554 ∂/∂th + 0.0646134478134985 ∂/∂ph
sage: p0[:]  # tol 1.0e-13
[1.29225788954106, 0.00438084990626460, 0.0189826106258554, 0.0646134478134985]
```

For instance, the components p_0^t and p_0^ϕ are recovered by:

```
sage: p0[0], p0[3]  # tol 1.0e-13
(1.29225788954106, 0.0646134478134985)
```

Let us check that the scalar square of p_0 is -1, i.e. is consistent with the mass parameter $\mu = 1$ used in the construction of the geodesic:

```
sage: g = M.metric()
sage: g.at(init_point)(p0, p0).subs(a=0.998)  # tol 1.0e-13
-1.00000000000000
```

The 4-momentum vector p at any value of the affine parameter λ, e.g. $\lambda = 200m$, is obtained by:

```
sage: p = geod.evaluate_tangent_vector(200); p
Tangent vector at Point on the Kerr spacetime M
sage: p in M.tangent_space(geod(200))
True
sage: p.display()  # tol 1.0e-13
1.316592599498746 ∂/∂t - 0.07370434215844164 ∂/∂r
 - 0.01091195426423706 ∂/∂th + 0.07600209768075264 ∂/∂ph
```

The particle mass μ computed at a given value of λ is returned by the method `evaluate_mu()`:

```
sage: geod.evaluate_mu(0)  # tol 1.0e-13
1.00000000000000
```

Of course, it should be conserved along the geodesic; actually it is, up to the numerical accuracy:

```
sage: geod.evaluate_mu(300)  # tol 1.0e-13
1.0000117978600134
```

Similarly, the conserved energy E, conserved angular momentum L and Carter constant Q are computed at any value of λ by respectively `evaluate_E()`, `evaluate_L()` and `evaluate_Q()`:

```
sage: geod.evaluate_E(0)  # tol 1.0e-13
0.883000000000000
sage: geod.evaluate_L(0)  # tol 1.0e-13
1.98200000000000
sage: geod.evaluate_Q(0)  # tol 1.0e-13
0.467000000000000
```

Let us check that the values of μ, E, L and Q evaluated at $\lambda = 300m$ are equal to those at $\lambda = 0$ up to the numerical accuracy of the integration scheme:

```
sage: geod.check_integrals_of_motion(300)  # tol 1.0e-13
  quantity         value            initial value       diff.      relative diff.
  $\mu^2$    1.0000235958592163   1.00000000000000    0.00002360     0.00002360
    $E$      0.883067996080701    0.883000000000000   0.00006800     0.00007701
    $L$       1.98248080818931    1.98200000000000    0.0004808      0.0002426
    $Q$      0.467214137649741    0.467000000000000   0.0002141      0.0004585
```

Decreasing the integration step leads to smaller errors:

```
sage: geod.integrate(step=0.001)
sage: geod.check_integrals_of_motion(300)  # tol 1.0e-13
  quantity         value            initial value       diff.      relative diff.
  $\mu^2$    1.0000047183936422   1.00000000000000     4.718e-6       4.718e-6
    $E$      0.883013604456676    0.883000000000000   0.00001360     0.00001541
    $L$      1.98209626120918     1.98200000000000    0.00009626     0.00004857
    $Q$      0.467042771975860    0.467000000000000   0.00004277     0.00009159
```

Various ways to initialize a geodesic

Instead of providing the integral of motions, as for `geod` above, one can initialize a geodesic by providing the Boyer-Lindquist components $(p_0^t, p_0^r, p_0^\theta, p_0^\phi)$ of the initial 4-momentum vector p_0. For instance:

```
sage: p0
Tangent vector p at Point P on the Kerr spacetime M
sage: p0[:]  # tol 1.0e-13
[1.29225788954106, 0.00438084990626460, 0.0189826106258554, 0.0646134478134985]
sage: geod2 = M.geodesic([0, 300], init_point, pt0=p0[0], pr0=p0[1],
....:                    pth0=p0[2], pph0=p0[3], a_num=0.998)
sage: geod2.initial_tangent_vector() == p0
True
```

As a check, we recover the same values of (μ, E, L, Q) as those that were used to initialize `geod`:

```
sage: geod2.evaluate_mu(0)
1.00000000000000
sage: geod2.evaluate_E(0)
0.883000000000000
sage: geod2.evaluate_L(0)
1.98200000000000
sage: geod2.evaluate_Q(0)
0.467000000000000
```

We may also initialize a geodesic by providing the mass μ and the three spatial components $(p_0^r, p_0^\theta, p_0^\phi)$ of the initial 4-momentum vector:

```
sage: geod3 = M.geodesic([0, 300], init_point, mu=1, pr0=p0[1],
....:                    pth0=p0[2], pph0=p0[3], a_num=0.998)
```

The component p_0^t is then automatically computed:

```
sage: geod3.initial_tangent_vector()[:]  # tol 1.0e-13
[1.29225788954106, 0.00438084990626460, 0.0189826106258554, 0.0646134478134985]
```

and we check the identity with the initial vector of `geod`, up to numerical errors:

```
sage: (geod3.initial_tangent_vector() - p0)[:]  # tol 1.0e-13
[2.22044604925031e-16, 0, 0, 0]
```

Another way to initialize a geodesic is to provide the conserved energy E, the conserved angular momentum L and the two components (p_0^r, p_0^θ) of the initial 4-momentum vector:

```
sage: geod4 = M.geodesic([0, 300], init_point, E=0.8830, L=1.982,
....:                    pr0=p0[1], pth0=p0[2], a_num=0.998)
sage: geod4.initial_tangent_vector()[:]
[1.29225788954106, 0.00438084990626460, 0.0189826106258554, 0.0646134478134985]
```

Again, we get a geodesic equivalent to `geod`:

```
sage: (geod4.initial_tangent_vector() - p0)[:]  # tol 1.0e-13
[0, 0, 0, 0]
```

check_integrals_of_motion(*affine_parameter*, *solution_key=None*)

Check the constancy of the four integrals of motion

INPUT:

- `affine_parameter` – value of the affine parameter λ
- `solution_key` – (default: `None`) string denoting the numerical solution to use for evaluating the various integrals of motion; if `None`, the latest solution computed by *`integrate()`* is used.

OUTPUT:

- a SageMath table with the absolute and relative differences with respect to the initial values.

evaluate_E(*affine_parameter*, *solution_key=None*)

Compute the conserved energy E at a given value of the affine parameter λ.

INPUT:

- `affine_parameter` – value of the affine parameter λ
- `solution_key` – (default: `None`) string denoting the numerical solution to use for the evaluation; if `None`, the latest solution computed by *`integrate()`* is used.

OUTPUT:

- value of E

evaluate_L(*affine_parameter*, *solution_key=None*)

Compute the conserved angular momentum about the rotation axis L at a given value of the affine parameter λ.

INPUT:

- `affine_parameter` – value of the affine parameter λ
- `solution_key` – (default: `None`) string denoting the numerical solution to use for the evaluation; if `None`, the latest solution computed by *`integrate()`* is used.

OUTPUT:

- value of L

evaluate_Q(*affine_parameter*, *solution_key=None*)

Compute the Carter constant Q at a given value of the affine parameter λ.

INPUT:

- `affine_parameter` – value of the affine parameter λ
- `solution_key` – (default: `None`) string denoting the numerical solution to use for the evaluation; if `None`, the latest solution computed by *integrate()* is used.

OUTPUT:

- value of Q

evaluate_mu(*affine_parameter*, *solution_key=None*)

Compute the mass parameter μ at a given value of the affine parameter λ.

INPUT:

- `affine_parameter` – value of the affine parameter λ
- `solution_key` – (default: `None`) string denoting the numerical solution to use for the evaluation; if `None`, the latest solution computed by *integrate()* is used.

OUTPUT:

- value of μ

evaluate_mu2(*affine_parameter*, *solution_key=None*)

Compute the square of the mass parameter μ^2 at a given value of the affine parameter λ.

INPUT:

- `affine_parameter` – value of the affine parameter λ
- `solution_key` – (default: `None`) string denoting the numerical solution to use for the evaluation; if `None`, the latest solution computed by *integrate()* is used.

OUTPUT:

- value of μ^2

evaluate_tangent_vector(*affine_parameter*, *solution_key=None*)

Return the tangent vector (4-momentum) at a given value of the affine parameter.

INPUT:

- `affine_parameter` – value of the affine parameter λ
- `solution_key` – (default: `None`) string denoting the numerical solution to use for the evaluation; if `None`, the latest solution computed by *integrate()* is used.

OUTPUT:

- instance of TangentVector representing the 4-momentum vector p at λ

initial_tangent_vector()

Return the initial 4-momentum vector.

OUTPUT:

- instance of TangentVector representing the initial 4-momentum p_0

EXAMPLES:

Initial 4-momentum vector of a null geodesic in Schwarzschild spacetime:

```
sage: from kerrgeodesic_gw import KerrBH
sage: M = KerrBH(0)
sage: BLchart = M.boyer_lindquist_coordinates()
sage: init_point = M((0, 6, pi/2, 0), name='P')
sage: geod = M.geodesic([0, 100], init_point, mu=0, E=1,
....:                   L=3, Q=0)
sage: p0 = geod.initial_tangent_vector(); p0
Tangent vector p at Point P on the Schwarzschild spacetime M
sage: p0.display()
p = 3/2 ∂/∂t + 1/6*sqrt(30) ∂/∂r + 1/12 ∂/∂ph
```

integrate(*step=None*, *method='odeint'*, *solution_key=None*, *parameters_values=None*, *verbose=False*, ***control_param*)

Solve numerically the geodesic equation.

INPUT:

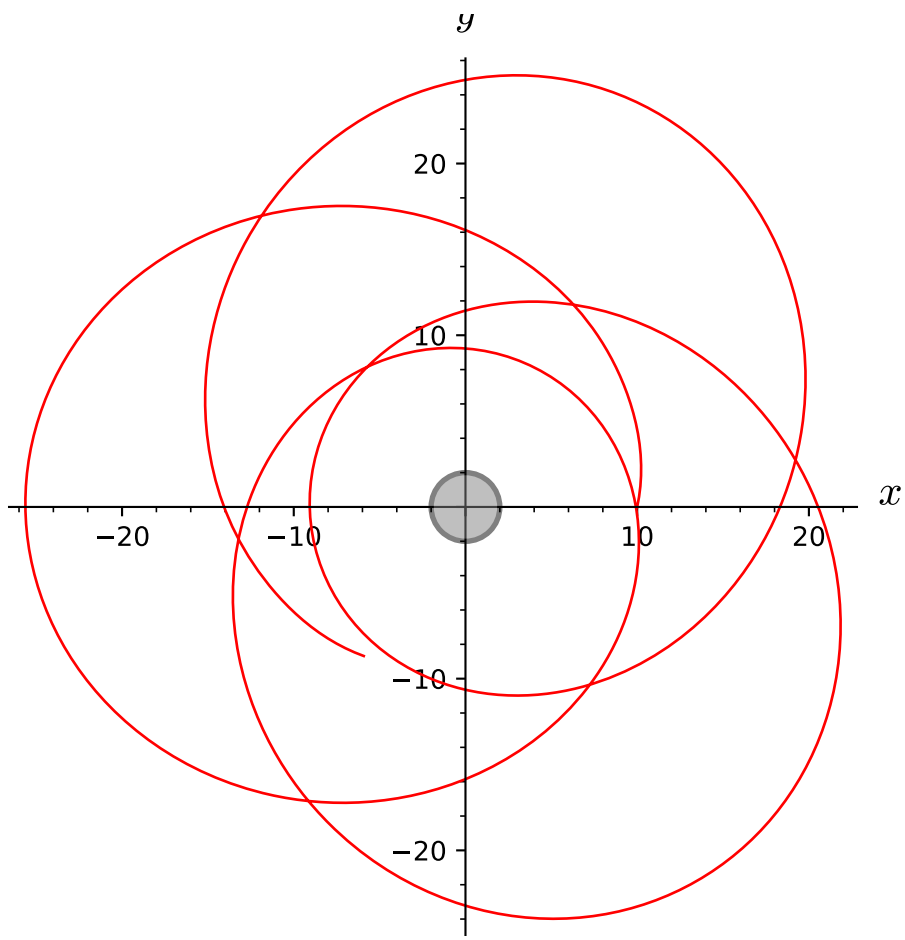
- `step` – (default: `None`) step $\delta\lambda$ for the integration, where λ is the affine parameter along the geodesic; default value is a hundredth of the range of λ declared when constructing the geodesic
- `method` – (default: `'odeint'`) numerical scheme to use for the integration; available algorithms are:
  - `'odeint'` - makes use of scipy.integrate.odeint via Sage solver `desolve_odeint()`; `odeint` invokes the LSODA algorithm of the ODEPACK suite, which automatically selects between implicit Adams method (for non-stiff problems) and a method based on backward differentiation formulas (BDF) (for stiff problems).
  - `'rk4_maxima'` - 4th order classical Runge-Kutta, which makes use of Maxima's dynamics package via Sage solver `desolve_system_rk4()` (quite slow)
  - `'dopri5'` - Dormand-Prince Runge-Kutta of order (4)5 provided by scipy.integrate.ode
  - `'dop853'` - Dormand-Prince Runge-Kutta of order 8(5,3) provided by scipy.integrate.ode

  and those provided by `GSL` via Sage class `ode_solver`:
  - `'rk2'` - embedded Runge-Kutta (2,3)
  - `'rk4'` - 4th order classical Runge-Kutta
  - `'rkf45'` - Runge-Kutta-Felhberg (4,5)
  - `'rkck'` - embedded Runge-Kutta-Cash-Karp (4,5)
  - `'rk8pd'` - Runge-Kutta Prince-Dormand (8,9)
  - `'rk2imp'` - implicit 2nd order Runge-Kutta at Gaussian points
  - `'rk4imp'` - implicit 4th order Runge-Kutta at Gaussian points
  - `'gear1'` - $M = 1$ implicit Gear
  - `'gear2'` - $M = 2$ implicit Gear
  - `'bsimp'` - implicit Bulirsch-Stoer (requires Jacobian)
- `solution_key` – (default: `None`) string to tag the numerical solution; if `None`, the string `method` is used.
- `parameters_values` – (default: `None`) list of numerical values of the parameters present in the system defining the geodesic, to be substituted in the equations before integration
- `verbose` – (default: `False`) prints information about the computation in progress
- `**control_param` – extra control parameters to be passed to the chosen solver

EXAMPLES:

Bound timelike geodesic in Schwarzschild spacetime:

```
sage: from kerrgeodesic_gw import KerrBH
sage: M = KerrBH(0)
sage: BLchart = M.boyer_lindquist_coordinates()
sage: init_point = M((0, 10, pi/2, 0), name='P')
sage: lmax = 1500.
sage: geod = M.geodesic([0, lmax], init_point, mu=1, E=0.973,
....:                   L=4.2, Q=0)
sage: geod.integrate()
sage: geod.plot(coordinates='xy')
Graphics object consisting of 2 graphics primitives
```



With the default integration step, the accuracy is not very good:

```
sage: geod.check_integrals_of_motion(lmax)  # tol 1.0e-13
  quantity          value             initial value       diff.     relative diff.
  $\mu^2$      1.000761704316941     1.00000000000000    0.0007617     0.0007617
    $E$       0.9645485805304451     0.973000000000000   -0.008451     -0.008686
    $L$        3.8897905637080923    4.20000000000000     -0.3102       -0.07386
    $Q$      5.673017835722329e-32          0           5.673e-32         -
```

Let us improve it by specifying a smaller integration step:

```
sage: geod.integrate(step=0.1)
sage: geod.check_integrals_of_motion(lmax)  # tol 1.0e-13
```

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```
quantity          value              initial value        diff.       relative diff.
$\mu^2$     1.0000101879128696     1.00000000000000    0.00001019      0.00001019
  $E$       0.9729448260004574     0.973000000000000   -0.00005517    -0.00005671
  $L$        4.197973829219027     4.20000000000000     -0.002026      -0.0004824
  $Q$      6.607560764960032e-32           0            6.608e-32          -
```

We may set the parameter `solution_key` to keep track of various numerical solutions:

```
sage: geod.integrate(step=0.1, solution_key='step_0.1')
sage: geod.integrate(step=0.02, solution_key='step_0.02')
```

and use it in the various evaluation functions:

```
sage: geod.evaluate_mu(lmax, solution_key='step_0.1')  # tol 1.0e-13
1.0000050939434606
sage: geod.evaluate_mu(lmax, solution_key='step_0.02')  # tol 1.0e-13
1.0000010212056811
```

plot(*coordinates='xyz', prange=None, solution_key=None, style='-', thickness=1, plot_points=1000, color='red', include_end_point=(True, True), end_point_offset=(0.001, 0.001), verbose=False, label_axes=None, plot_horizon=True, horizon_color='black', fill_BH_region=True, BH_region_color='grey', display_tangent=False, color_tangent='blue', plot_points_tangent=10, width_tangent=1, scale=1, aspect_ratio='automatic', **kwds*)

Plot the geodesic in terms of the coordinates (t, x, y, z) deduced from the Boyer-Lindquist coordinates (t, r, θ, ϕ) via the standard polar to Cartesian transformations.

INPUT:

- `coordinates` – (default: `'xyz'`) string indicating which of the coordinates (t, x, y, z) to use for the plot
- `prange` – (default: `None`) range of the affine parameter λ for the plot; if `None`, the entire range declared during the construction of the geodesic is considered
- `solution_key` – (default: `None`) string denoting the numerical solution to use for the plot; if `None`, the latest solution computed by *`integrate()`* is used.
- `verbose` – (default: `False`) determines whether information is printed about the plotting in progress
- `plot_horizon` – (default: `True`) determines whether the black hole event horizon is drawn
- `horizon_color` – (default: `'black'`) color of the event horizon
- `fill_BH_region` – (default: `True`) determines whether the black hole region is colored (for 2D plots only)
- `BH_region_color` – (default: `'grey'`) color of the event horizon
- `display_tangent` – (default: `False`) determines whether some tangent vectors should also be plotted
- `color_tangent` – (default: `blue`) color of the tangent vectors when these are plotted
- `plot_points_tangent` – (default: 10) number of tangent vectors to display when these are plotted
- `width_tangent` – (default: 1) sets the width of the arrows representing the tangent vectors
- `scale` – (default: 1) scale applied to the tangent vectors before displaying them

See also:

DifferentiableCurve.plot for the other input parameters

OUTPUT:

- either a 2D graphic oject (2 coordinates specified in the parameter `coordinates`) or a 3D graphic object (3 coordinates in `coordinates`)

CHAPTER

THREE

SPHERICAL AND SPHEROIDAL HARMONICS

3.1 Spin-weighted spherical harmonics

Spin-weighted spherical harmonics ${}_sY_l^m(\theta,\phi)$

`kerrgeodesic_gw.spin_weighted_spherical_harm.`**`spin_weighted_spherical_harmonic`**(*s*, *l*, *m*, *theta*, *phi*, *condon_shortley=True*, *cached=True*, *numerical=None*)

Return the spin-weighted spherical harmonic of spin weight `s` and indices `(l,m)`.

INPUT:

- `s` – integer; the spin weight
- `l` – non-negative integer; the harmonic degree
- `m` – integer within the range `[-l, l]`; the azimuthal number
- `theta` – colatitude angle
- `phi` – azimuthal angle
- `condon_shortley` – (default: `True`) determines whether the Condon-Shortley phase of $(-1)^m$ is taken into account (see below)
- `cached` – (default: `True`) determines whether the result shall be cached; setting `cached` to `False` forces a new computation, without caching the result
- `numerical` – (default: `None`) determines whether a symbolic or a numerical computation of a given type is performed; allowed values are
  - `None`: the type of computation is deduced from the type of `theta`
  - `RDF`: Sage's machine double precision floating-point numbers (`RealDoubleField`)
  - `RealField(n)`, where `n` is a number of bits: Sage's floating-point numbers with an arbitrary precision; note that `RR` is a shortcut for `RealField(53)`.
  - `float`: Python's floating-point numbers

OUTPUT:

- the value of ${}_sY_l^m(\theta,\phi)$, either as a symbolic expression or as floating-point complex number of the type determined by `numerical`

ALGORITHM:

The spin-weighted spherical harmonic is evaluated according to Eq. (3.1) of J. N. Golberg et al., J. Math. Phys. **8**, 2155 (1967) [doi:10.1063/1.1705135], with an extra $(-1)^m$ factor (the so-called *Condon-Shortley phase*) if `condon_shortley` is `True`, the actual formula being then the one given in Wikipedia article Spin-weighted_spherical_harmonics#Calculating

EXAMPLES:

```
sage: from kerrgeodesic_gw import spin_weighted_spherical_harmonic
sage: theta, phi = var('theta phi')
sage: spin_weighted_spherical_harmonic(-2, 2, 1, theta, phi)
1/4*(sqrt(5)*cos(theta) + sqrt(5))*e^(I*phi)*sin(theta)/sqrt(pi)
sage: spin_weighted_spherical_harmonic(-2, 2, 1, theta, phi,
....:                                  condon_shortley=False)
-1/4*(sqrt(5)*cos(theta) + sqrt(5))*e^(I*phi)*sin(theta)/sqrt(pi)
sage: spin_weighted_spherical_harmonic(-2, 2, 1, pi/3, pi/4)
(3/32*I + 3/32)*sqrt(5)*sqrt(3)*sqrt(2)/sqrt(pi)
```

Evaluation as floating-point numbers: the type of the output is deduced from the input:

```
sage: spin_weighted_spherical_harmonic(-2, 2, 1, 1.0, 2.0)  # tol 1.0e-13
-0.170114676286891 + 0.371707349012686*I
sage: parent(_)
Complex Field with 53 bits of precision
sage: spin_weighted_spherical_harmonic(-2, 2, 1, RDF(2.0), RDF(3.0))  # tol 1.0e-13
-0.16576451879564585 + 0.023629159118690464*I
sage: parent(_)
Complex Double Field
sage: spin_weighted_spherical_harmonic(-2, 2, 1, float(3.0), float(4.0))  # tol 1.0e-13
(-0.0002911423884400524-0.00033709085352998027j)
sage: parent(_)
<type 'complex'>
```

Computation with arbitrary precision are possible (here 100 bits):

```
sage: R100 = RealField(100); R100
Real Field with 100 bits of precision
sage: spin_weighted_spherical_harmonic(-2, 2, 1, R100(1.5), R100(2.0))  # tol 1.0e-28
-0.14018136537676185317636108802 + 0.30630187143465275236861476906*I
```

Even when the entry is symbolic, numerical evaluation can be enforced via the argument `numerical`. For instance, setting `numerical` to `RDF` (SageMath's Real Double Field):

```
sage: spin_weighted_spherical_harmonic(-2, 2, 1, pi/3, pi/4, numerical=RDF)  # tol 1.0e-13
0.2897056515173923 + 0.28970565151739225*I
sage: parent(_)
Complex Double Field
```

One can also use `numerical=RR` (SageMath's Real Field with precision set to 53 bits):

```
sage: spin_weighted_spherical_harmonic(-2, 2, 1, pi/3, pi/4, numerical=RR)   # tol 1.0e-13
0.289705651517392 + 0.289705651517392*I
sage: parent(_)
Complex Field with 53 bits of precision
```

Another option is to use Python floats:

```
sage: spin_weighted_spherical_harmonic(-2, 2, 1, pi/3, pi/4, numerical=float)  # tol 1.0e-13
(0.28970565151739225+0.2897056515173922j)
sage: parent(_)
<type 'complex'>
```

One can go beyond double precision, for instance using 100 bits of precision:

```
sage: spin_weighted_spherical_harmonic(-2, 2, 1, pi/3, pi/4,
....:                                  numerical=RealField(100))  # tol 1.0e-28
0.28970565151739218525664455148 + 0.28970565151739218525664455148*I
sage: parent(_)
Complex Field with 100 bits of precision
```

3.2 Spin-weighted spheroidal harmonics

Spin-weighted spheroidal harmonics

`kerrgeodesic_gw.spin_weighted_spheroidal_harm.`**`spin_weighted_spheroidal_eigenvalue`**(*s*, *l*, *m*, *gamma*, *verbose=False*, *cached=True*, *min_nmax=8*)

Return the spin-weighted oblate spheroidal eigenvalue of spin weight `s`, degree `l`, azimuthal order `m` and spheroidicity `gamma`.

INPUT:

- `s` – integer; the spin weight
- `l` – non-negative integer; the harmonic degree
- `m` – integer within the range `[-l, l]`; the azimuthal number
- `gamma` – spheroidicity parameter γ
- `verbose` – (default: `False`) determines whether some details of the computation are printed out
- `cached` – (default: `True`) determines whether the eigenvalue and the eigenvectors shall be cached; setting `cached` to `False` forces a new computation, without caching the result
- `min_nmax` – (default: 8) integer; floor for the evaluation of the parameter `nmax`, which sets the highest degree of the spherical harmonic expansion as `l+nmax`.

OUTPUT:

- eigenvalue λ related to the eigenvalue $\mathcal{E}_{\ell m}$ of the spheroidal harmonic by

$$\lambda = \mathcal{E}_{\ell m} - 2m\gamma + \gamma^2 - s(s+1)$$

ALGORITHM:

The method is adapted from that exposed in Appendix A of S.A. Hughes, Phys. Rev. D **61**, 084004 (2000) [doi:10.1103/PhysRevD.61.084004].

EXAMPLES:

```
sage: from kerrgeodesic_gw import spin_weighted_spheroidal_eigenvalue
sage: spin_weighted_spheroidal_eigenvalue(-2, 2, 1, 1.2)  # tol 1.0e-13
0.5167945263162421
sage: spin_weighted_spheroidal_eigenvalue(-2, 2, 1, 1.2, cached=False)  # tol 1.0e-13
0.5167945263162421
sage: spin_weighted_spheroidal_eigenvalue(-2, 2, 1, 0)
4.0
```

kerrgeodesic_gw.spin_weighted_spheroidal_harm.**spin_weighted_spheroidal_harmonic**(*s*, *l*, *m*, *gamma*, *theta*, *phi*, *verbose=False*, *cached=True*, *min_nmax=8*)

Return the spin-weighted oblate spheroidal harmonic of spin weight `s`, degree `l`, azimuthal order `m` and spheroidicity `gamma`.

INPUT:

- `s` – integer; the spin weight
- `l` – non-negative integer; the harmonic degree
- `m` – integer within the range `[-l, l]`; the azimuthal number
- `gamma` – spheroidicity parameter
- `theta` – colatitude angle
- `phi` – azimuthal angle
- `verbose` – (default: `False`) determines whether some details of the computation are printed out
- `cached` – (default: `True`) determines whether the eigenvectors shall be cached; setting `cached` to `False` forces a new computation, without caching the result
- `min_nmax` – (default: 8) integer; floor for the evaluation of the parameter `nmax`, which sets the highest degree of the spherical harmonic expansion as `l+nmax`.

OUTPUT:

- value of ${}_sS^{\gamma}_{lm}(\theta,\phi)$

ALGORITHM:

The spin-weighted oblate spheroidal harmonics are computed by an expansion over spin-weighted *spherical* harmonics, the coefficients of the expansion being obtained by solving an eigenvalue problem, as exposed in Appendix A of S.A. Hughes, Phys. Rev. D **61**, 084004 (2000) [doi:10.1103/PhysRevD.61.084004].

EXAMPLES:

```
sage: from kerrgeodesic_gw import spin_weighted_spheroidal_harmonic
sage: spin_weighted_spheroidal_harmonic(-2, 2, 2, 1.1, pi/2, 0)  # tol 1.0e-13
0.08702532727529422
sage: spin_weighted_spheroidal_harmonic(-2, 2, 2, 1.1, pi/3, pi/3)  # tol 1.0e-13
-0.14707166027821453 + 0.25473558795537715*I
sage: spin_weighted_spheroidal_harmonic(-2, 2, 2, 1.1, pi/3, pi/3, cached=False)   # tol 1.0e-13
-0.14707166027821453 + 0.25473558795537715*I
sage: spin_weighted_spheroidal_harmonic(-2, 2, 2, 1.1, pi/3, pi/4)  # tol 1.0e-13
1.801108380050024e-17 + 0.2941433205564291*I
sage: spin_weighted_spheroidal_harmonic(-2, 2, -1, 1.1, pi/3, pi/3, cached=False)  # tol 1.0e-13
0.11612826056899399 - 0.20114004750009495*I
```

Test that the relation ${}_sS^{\gamma}_{lm}(\theta,\phi) = (-1)^{l+s}\,{}_sS^{-\gamma}_{l,-m}(\pi-\theta,-\phi)$ [cf. Eq. (2.3) of Arxiv 1810.00432], which is used to evaluate ${}_sS^{\gamma}_{lm}(\theta,\phi)$ when $m<0$ and `cached` is `True`, is correctly implemented:

```
sage: spin_weighted_spheroidal_harmonic(-2, 2, -2, 1.1, pi/3, pi/3)  # tol 1.0e-13
-0.04097260436590737 - 0.07096663248016997*I
sage: abs(_ - spin_weighted_spheroidal_harmonic(-2, 2, -2, 1.1, pi/3, pi/3,
....:                                           cached=False)) < 1.e-13
True
```

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```
sage: spin_weighted_spheroidal_harmonic(-2, 3, -1, 1.1, pi/3, pi/3)  # tol 1.0e-13
0.1781880511506843 - 0.3086307578946672*I
sage: abs(_ - spin_weighted_spheroidal_harmonic(-2, 3, -1, 1.1, pi/3, pi/3,
....:                                           cached=False)) < 1.e-13
True
```

CHAPTER

FOUR

AMPLITUDE FACTORS $Z^{\infty}_{\ell M}(R)$

Functions $Z^{\infty}_{\ell m}(r)$ giving the amplitude of the gravitational radiation emitted by a particle on a circular orbit at radius r about a Kerr black hole.

REFERENCES:

- S. A. Teukolsky, Astrophys. J. **185**, 635 (1973)
- S. Detweiler, Astrophys. J. **225**, 687 (1978)
- M. Shibata, Phys. Rev. D **50**, 6297 (1994)
- D. Kennefick, Phys. Rev. D **58**, 064012 (1998)
- S. A. Hughes, Phys. Rev. D **61**, 084004 (2000) [doi:10.1103/PhysRevD.61.084004]
- E. Gourgoulhon, A. Le Tiec, F. Vincent, N. Warburton, Astron. Astrophys. **627**, A92 (2019) [doi:10.1051/0004-6361/201935406]; Arxiv 1903.02049

`kerrgeodesic_gw.zinf.Zinf`(*a*, *l*, *m*, *r*, *algorithm='spline'*)

Amplitude factor of the mode (ℓ, m).

The factor $Z^{\infty}_{\ell m}(r)$ is obtained by spline interpolation of tabulated numerical solutions of the radial component of the Teukolsky equation.

INPUT:

- `a` – BH angular momentum parameter (in units of M, the BH mass)
- `l` – integer >= 2; the harmonic degree ℓ
- `m` – integer within the range `[-l, l]`; the azimuthal number m
- `r` – areal radius of the orbit (in units of M)
- `algorithm` – (default: `'spline'`) string describing the computational method; allowed values are
  - `'spline'`: spline interpolation of tabulated data
  - `'1.5PN'` (only for `a=0`): 1.5-post-Newtonian expansion following E. Poisson, Phys. Rev. D **47**, 1497 (1993) [doi:10.1103/PhysRevD.47.1497], with a minus one factor accounting for a different convention for the metric signature.

OUTPUT:

- coefficient $Z^{\infty}_{\ell m}(r)$ (in units of M^{-2})

EXAMPLES:

```
sage: from kerrgeodesic_gw import Zinf
sage: Zinf(0.98, 2, 2, 1.7)  # tol 1.0e-13
-0.04302234478778856 + 0.28535368610053824*I
sage: Zinf(0., 2, 2, 10.)  # tol 1.0e-13
0.0011206407919254163 - 0.0003057608384581628*I
sage: Zinf(0., 2, 2, 10., algorithm='1.5PN')  # tol 1.0e-13
0.0011971529546749354 - 0.0003551610880408921*I
```

`kerrgeodesic_gw.zinf.`**`Zinf_Schwarzchild_PN`**(*l*, *m*, *r*)

Amplitude factor of the mode (ℓ, m) for a Schwarzschild BH at the 1.5PN level.

The 1.5PN formulas are taken from E. Poisson, Phys. Rev. D **47**, 1497 (1993), doi:10.1103/PhysRevD.47.1497.

INPUT:

- `l` – integer >= 2; the harmonic degree ℓ
- `m` – integer within the range `[-l, l]`; the azimuthal number m
- `r` – areal radius of the orbit (in units of M, the BH mass)

OUTPUT:

- coefficient $Z^{\infty}_{\ell m}(r)$ (in units of M^{-2})

EXAMPLES:

```
sage: from kerrgeodesic_gw import Zinf_Schwarzchild_PN
sage: Zinf_Schwarzchild_PN(2, 2, 6.)  # tol 1.0e-13
-0.00981450418730346 + 0.003855681972781947*I
sage: Zinf_Schwarzchild_PN(5, 3, 6.)  # tol 1.0e-13
-6.958527913913504e-05*I
```

CHAPTER

FIVE

GRAVITATIONAL RADIATION BY A SINGLE PARTICLE

Gravitational radiation by a particle on a circular orbit around a Kerr black hole

The gravitational wave emitted by a particle of mass μ on a circular orbit in the equatorial plane of a Kerr black hole of mass M and angular momentum parameter a is given by the formula:

$$h_+ - ih_\times = \frac{2\mu}{r} \sum_{\ell=2}^{\infty} \sum_{\substack{m=-\ell \\ m\neq 0}}^{\ell} \frac{Z_{\ell m}^{\infty}(r_0)}{(m\omega_0)^2} \, {}_{-2}S_{\ell m}^{am\omega_0}(\theta,\phi)\, e^{-im\phi_0} e^{-im\omega_0(t-r_*)} \tag{5.1}$$

where

- $h_+ = h_+(t,r,\theta,\phi)$ and $h_\times = h_\times(t,r,\theta,\phi)$, (t,r,θ,ϕ) being the Boyer-Lindquist coordinates of the observer
- r_* is the tortoise coordinate corresponding to r
- r_0 is the Boyer-Lindquist radius of the particle's orbit
- ϕ_0 is some constant phase factor
- ω_0 is the orbital angular velocity
- $Z_{\ell m}^{\infty}(r_0)$ is a solution of the radial component of the Teukolsky equation (cf. `Zinf()`)
- ${}_{-2}S_{\ell m}^{am\omega_0}(\theta,\phi)$ is the spin-weighted spheroidal harmonic of weight -2 (cf. `spin_weighted_spheroidal_harmonic()`)

According to Eq. (5.1), the Fourier-series expansion of the waveform $(h_+, h_\times)$ received at the location (t,r,θ,ϕ) is

$$h_{+,\times}(t,r,\theta,\phi) = \sum_{m=1}^{+\infty} \left[A_m^{+,\times}(r,\theta)\cos(m\psi) + B_m^{+,\times}(r,\theta)\sin(m\psi) \right], \tag{5.2}$$

where

$$\psi := \omega_0(t-r_*) - \phi + \phi_0,$$

ω_0 being the orbital angular velocity of the particle and r_* the tortoise coordinate corresponding to r.

Note that the dependence of the Fourier coefficients $A_m^{+,\times}(r,\theta)$ and $B_m^{+,\times}(r,\theta)$ with respect to r is simply μ/r, where μ is the particle's mass, i.e. we may consider the following rescaled Fourier coefficients, which depend on θ only:

$$\bar{A}_m^{+,\times}(\theta) := \frac{r}{\mu} A_m^{+,\times}(r,\theta) \quad \text{and} \quad \bar{B}_m^{+,\times}(\theta) := \frac{r}{\mu} B_m^{+,\times}(r,\theta)$$

According to Eqs. (5.1) and (5.2), we have

$$\bar{A}_m^{+}(\theta) = \frac{2}{(m\omega_0)^2} \sum_{\ell=2}^{\infty} \mathrm{Re}\left(Z_{\ell m}^{\infty}(r_0)\right) \left[(-1)^\ell \, {}_{-2}S_{\ell,-m}^{-am\omega_0}(\theta,0) + {}_{-2}S_{\ell m}^{am\omega_0}(\theta,0) \right] \tag{5.3}$$

$$\bar{B}^{+}_{m}(\theta) = \frac{2}{(m\omega_0)^2} \sum_{\ell=2}^{\infty} \mathrm{Im}\left(Z^{\infty}_{\ell m}(r_0)\right) \left[(-1)^{\ell}\, {}_{-2}S^{-am\omega_0}_{\ell,-m}(\theta,0) + {}_{-2}S^{am\omega_0}_{\ell m}(\theta,0)\right] \tag{5.4}$$

$$\bar{A}^{\times}_{m}(\theta) = \frac{2}{(m\omega_0)^2} \sum_{\ell=2}^{\infty} \mathrm{Im}\left(Z^{\infty}_{\ell m}(r_0)\right) \left[(-1)^{\ell}\, {}_{-2}S^{-am\omega_0}_{\ell,-m}(\theta,0) - {}_{-2}S^{am\omega_0}_{\ell m}(\theta,0)\right] \tag{5.5}$$

$$\bar{B}^{\times}_{m}(\theta) = \frac{2}{(m\omega_0)^2} \sum_{\ell=2}^{\infty} \mathrm{Re}\left(Z^{\infty}_{\ell m}(r_0)\right) \left[(-1)^{\ell+1}\, {}_{-2}S^{-am\omega_0}_{\ell,-m}(\theta,0) + {}_{-2}S^{am\omega_0}_{\ell m}(\theta,0)\right] \tag{5.6}$$

This module implements the following functions:

- `h_plus_particle()`: evaluates rh_+/μ via Eq. (5.2)
- `h_cross_particle()`: evaluates $rh_\times/\mu$ via Eq. (5.2)
- `h_particle_quadrupole()`: evaluates rh_+/μ or $rh_\times/\mu$ at the quadrupole approximation
- `h_plus_particle_fourier()`: evaluates rA^+_m/μ and rB^+_m/μ via Eqs. (5.3)-(5.4)
- `h_cross_particle_fourier()`: evaluates $rA^\times_m/\mu$ and $rB^\times_m/\mu$ via Eqs. (5.5)-(5.6)
- `h_amplitude_particle_fourier()`: evaluates $(r/\mu)\sqrt{(A^+_m)^2 + (B^+_m)^2}$ and $(r/\mu)\sqrt{(A^\times_m)^2 + (B^\times_m)^2}$
- `h_particle_signal()`: time sequence of rh_+/μ or $rh_\times/\mu$
- `plot_h_particle()`: plot rh_+/μ and/or $rh_\times/\mu$ in terms of the retarded time
- `plot_spectrum_particle()`: plot $(r/\mu)\sqrt{(A^{+,\times}_m)^2 + (B^{+,\times}_m)^2}$ in terms of m
- `radiated_power_particle()`: total radiated power (gravitational luminosity)
- `secular_frequency_change()`: change in orbital frequency $\dot{\omega}_0/\omega_0$ due to the gravitational radiation reaction
- `decay_time()`: time for the orbit to shrink to an orbit of given radius

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`kerrgeodesic_gw.gw_particle.`**`decay_time`**(*a*, *r_init*, *r_final*, *l_max=None*, *m_min=1*, *approximation=None*, *quad_epsrel=1e-06*, *quad_limit=500*)

Return the time spent in the migration from a circular orbit of radius `r_init` to that of radius `r_final`, induced by gravitational radiation reaction.

INPUT:

- `a` – BH angular momentum parameter (in units of M, the BH mass)
- `r_init` – Boyer-Lindquist radius of the initial orbit (in units of M)
- `r_final` – Boyer-Lindquist radius of the final orbit (in units of M)
- `l_max` – (default: `None`) upper bound in the summation over the harmonic degree ℓ; if `None`, `l_max` is determined automatically from the available tabulated data
- `m_min` – (default: 1) lower bound in the summation over the Fourier mode m

- `approximation` – (default: `None`) string describing the computational method; allowed values are
  - `None`: exact computation
  - `'1.5PN'` (only for `a=0`): 1.5-post-Newtonian expansion following E. Poisson, Phys. Rev. D **47**, 1497 (1993) [doi:10.1103/PhysRevD.47.1497]
  - `'quadrupole'` (only for `a=0`): quadrupole approximation (0-post-Newtonian); see *h_particle_quadrupole()*
- `quad_epsrel` – (default: `1.e-6`) relative error tolerance in the computation of the integral giving the decay time
- `quad_limit` – (default: `500`) upper bound on the number of subintervals used in the adaptive algorithm to compute the integral (this corresponds to the argument `limit` of SciPy's function quad)

OUTPUT:

- rescaled decay time $T(\mu/M^2)$, where M is the BH mass and μ the mass of the orbiting particle.

EXAMPLES:

Time to migrate from $r_0 = 10M$ to $r_0 = 6M$ around a Schwarzschild black hole:

```
sage: from kerrgeodesic_gw import decay_time
sage: decay_time(0, 10, 6)  # tol 1.0e-13
90.80876605028857
```

Let us check that at large radius, there is a good agreement with the quadrupole formula:

```
sage: decay_time(0, 50, 6)  # tol 1.0e-13
121893.29664651724
sage: decay_time(0, 50, 6, approximation='quadrupole')  # tol 1.0e-13
122045.0
```

`kerrgeodesic_gw.gw_particle.`**`h_amplitude_particle_fourier`**(*m, a, r0, theta, l_max=10, algorithm_Zinf='spline'*)

Return the amplitude Fourier mode of a given order m of the rescaled gravitational wave emitted by a particle in circular orbit around a Kerr black hole.

The rescaled Fourier mode of order m received at the location (t, r, θ, ϕ) is

$$\frac{r}{\mu} h_m^{+,\times} = \bar{A}_m^{+,\times} \cos(m\psi) + \bar{B}_m^{+,\times} \sin(m\psi)$$

where μ is the particle mass and $\psi := \omega_0(t - r_*) - \phi$, ω_0 being the orbital frequency of the particle and r_* the tortoise coordinate corresponding to r and $\bar{A}_m^{+,\times}$ and $\bar{B}_m^{+,\times}$ are given by Eqs. (5.3)-(5.6) above.

The $+$ and $\times$ amplitudes of the Fourier mode m are defined respectively by

$$\frac{r}{\mu}|h_m^+| := \sqrt{(\bar{A}_m^+)^2 + (\bar{B}_m^+)^2} \quad \text{and} \quad \frac{r}{\mu}|h_m^\times| := \sqrt{(\bar{A}_m^\times)^2 + (\bar{B}_m^\times)^2}$$

INPUT:

- `m` – positive integer defining the Fourier mode
- `a` – BH angular momentum parameter (in units of M, the BH mass)
- `r0` – Boyer-Lindquist radius of the orbit (in units of M)
- `theta` – Boyer-Lindquist colatitute θ of the observer
- `l_max` – (default: 10) upper bound in the summation over the harmonic degree ℓ in Eqs. (5.3)-(5.6)

- `algorithm_Zinf` – (default: `'spline'`) string describing the computational method for $Z^{\infty}_{\ell m}(r_0)$; allowed values are
  - `'spline'`: spline interpolation of tabulated data
  - `'1.5PN'` (only for `a=0`): 1.5-post-Newtonian expansion following E. Poisson, Phys. Rev. D **47**, 1497 (1993) [doi:10.1103/PhysRevD.47.1497]

OUTPUT:

- tuple $((r/\mu)|h^{+}_{m}|,\ (r/\mu)|h^{\times}_{m}|)$ (cf. the above expression)

EXAMPLE:

For a Schwarzschild black hole ($a = 0$):

```
sage: from kerrgeodesic_gw import h_amplitude_particle_fourier
sage: a = 0
sage: h_amplitude_particle_fourier(2, a, 6., pi/2)  # tol 1.0e-13
(0.27875846152963557, 1.5860176188287866e-16)
sage: h_amplitude_particle_fourier(2, a, 6., pi/4)  # tol 1.0e-13
(0.47180033963220214, 0.45008696580919527)
sage: h_amplitude_particle_fourier(2, a, 6., 0)  # tol 1.0e-13
(0.6724377101568336, 0.6724377101568336)
sage: h_amplitude_particle_fourier(2, a, 6., pi/4, l_max=5)  # tol 1.0e-13
(0.47179830286565255, 0.4500948389153302)
sage: h_amplitude_particle_fourier(2, a, 6., pi/4, l_max=5,  # tol 1.0e-13
....:                              algorithm_Zinf='1.5PN')
(0.5381495951380861, 0.5114366815383188)
```

For a rapidly rotating Kerr black hole ($a = 0.95M$):

```
sage: a = 0.95
sage: h_amplitude_particle_fourier(2, a, 6., pi/4)  # tol 1.0e-13
(0.39402068296301823, 0.37534143024659444)
sage: h_amplitude_particle_fourier(2, a, 2., pi/4)  # tol 1.0e-13
(0.7358730645589858, 0.7115113031184368)
```

`kerrgeodesic_gw.gw_particle.`**`h_cross_particle`**(*a*, *r0*, *u*, *theta*, *phi*, *phi0=0*, *l_max=10*, *m_min=1*, *algorithm_Zinf='spline'*)

Return the rescaled $h_{\times}$-part of the gravitational radiation emitted by a particle in circular orbit around a Kerr black hole.

The computation is based on Eq. (5.2) above.

INPUT:

- `a` – BH angular momentum parameter (in units of M, the BH mass)
- `r0` – Boyer-Lindquist radius of the orbit (in units of M)
- `u` – retarded time coordinate of the observer (in units of M): $u = t - r_*$, where t is the Boyer-Lindquist time coordinate and r_* is the tortoise coordinate
- `theta` – Boyer-Lindquist colatitute θ of the observer
- `phi` – Boyer-Lindquist azimuthal coordinate ϕ of the observer
- `phi0` – (default: 0) phase factor
- `l_max` – (default: 10) upper bound in the summation over the harmonic degree ℓ
- `m_min` – (default: 1) lower bound in the summation over the Fourier mode m
- `algorithm_Zinf` – (default: `'spline'`) string describing the computational method for $Z^{\infty}_{\ell m}(r_0)$; allowed values are

- `'spline'`: spline interpolation of tabulated data
- `'1.5PN'` (only for `a=0`): 1.5-post-Newtonian expansion following E. Poisson, Phys. Rev. D **47**, 1497 (1993) [doi:10.1103/PhysRevD.47.1497]

OUTPUT:

- the rescaled waveform $(r/\mu)h_\times$, where μ is the particle's mass and r is the Boyer-Lindquist radial coordinate of the observer

EXAMPLES:

Let us consider the case $a = 0$ (Schwarzschild black hole) and $r_0 = 6M$ (emission from the ISCO). For $\theta = \pi/2$, we have $h_\times = 0$:

```
sage: from kerrgeodesic_gw import h_cross_particle
sage: a = 0
sage: h_cross_particle(a, 6., 0., pi/2, 0.)  # tol 1.0e-13
1.0041370414185673e-16
```

while for $\theta = \pi/4$, we have:

```
sage: h_cross_particle(a, 6., 0., pi/4, 0.)  # tol 1.0e-13
0.275027796440582
sage: h_cross_particle(a, 6., 0., pi/4, 0., l_max=5)  # tol 1.0e-13
0.2706516303570341
sage: h_cross_particle(a, 6., 0., pi/4, 0., l_max=5, algorithm_Zinf='1.5PN')  # tol 1.0e-13
0.2625307460899205
```

For an orbit of larger radius ($r_0 = 12M$), the 1.5-post-Newtonian approximation is in better agreement with the exact computation:

```
sage: h_cross_particle(a, 12., 0., pi/4, 0.)
0.1050751824554463
sage: h_cross_particle(a, 12., 0., pi/4, 0., l_max=5, algorithm_Zinf='1.5PN')  # tol 1.0e-13
0.10244926162224487
```

A plot of the waveform generated by a particle orbiting at the ISCO:

```
sage: hc = lambda u: h_cross_particle(a, 6., u, pi/4, 0.)
sage: plot(hc, (0, 200.), axes_labels=[r'$(t-r_*)/M$', r'$r h_\times/\mu$'],
....:      gridlines=True, frame=True, axes=False)
Graphics object consisting of 1 graphics primitive
```

Case $a/M = 0.95$ (rapidly rotating Kerr black hole):

```
sage: a = 0.95
sage: h_cross_particle(a, 2., 0., pi/4, 0.)  # tol 1.0e-13
-0.2681353673743396
```

Assessing the importance of the mode $m = 1$:

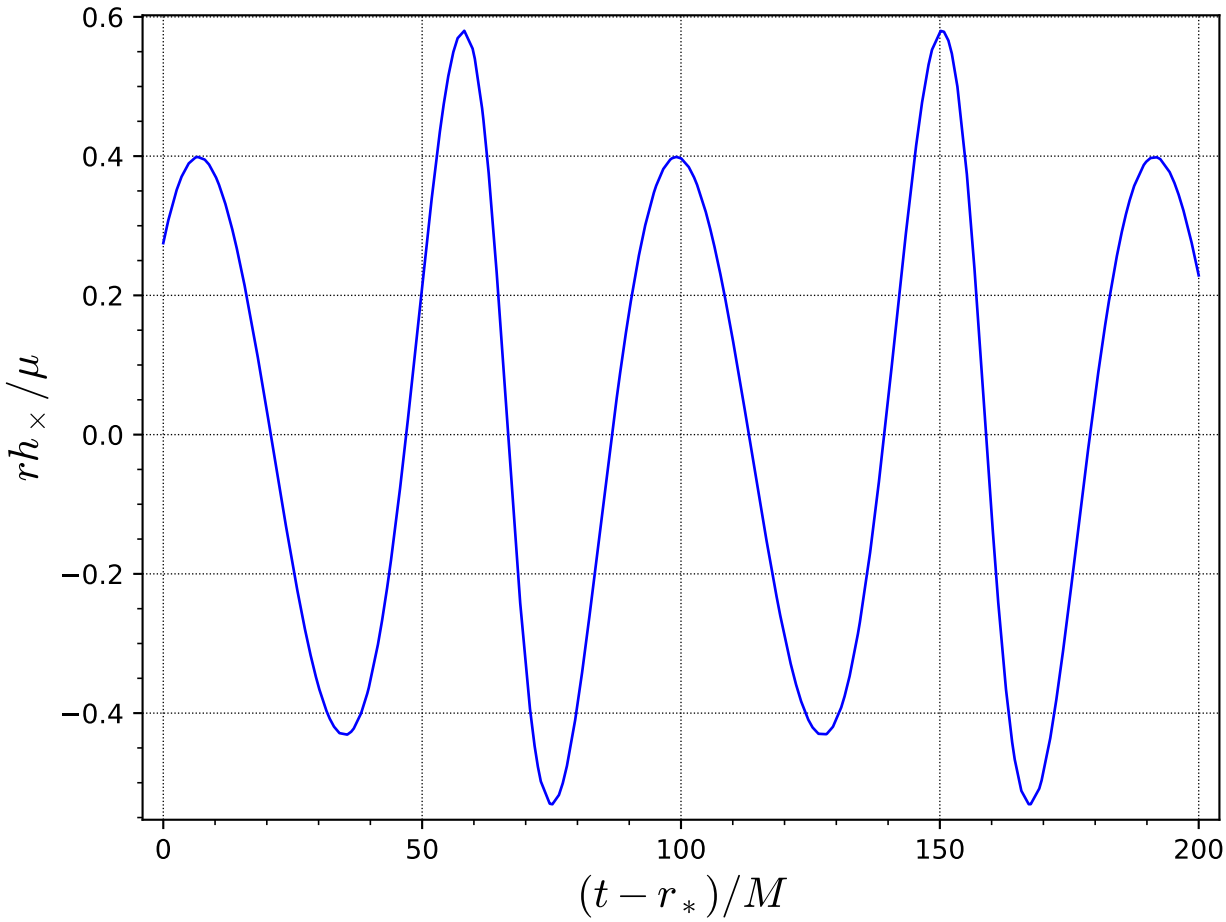
```
sage: h_cross_particle(a, 2., 0., pi/4, 0., m_min=2)  # tol 1.0e-13
-0.3010579420748449
```

`kerrgeodesic_gw.gw_particle.`**`h_cross_particle_fourier`**(*m*, *a*, *r0*, *theta*, *l_max=10*, *algorithm_Zinf='spline'*)

Return the Fourier mode of a given order m of the rescaled $h_\times$-part of the gravitational wave emitted by a particle in circular orbit around a Kerr black hole.

The rescaled Fourier mode of order m received at the location (t, r, θ, ϕ) is

$$\frac{r}{\mu} h_m^\times = \bar{A}_m^\times \cos(m\psi) + \bar{B}_m^\times \sin(m\psi)$$


0.6
0.4
0.2
0.0
−0.2
−0.4
$rh_\times/\mu$
0
50
100
150
200
$(t-r_*)/M$

where μ is the particle mass and $\psi := \omega_0(t - r_*) - \phi$, ω_0 being the orbital frequency of the particle and r_* the tortoise coordinate corresponding to r and $\bar{A}^{\times}_m$ and $\bar{B}^{\times}_m$ are given by Eqs. (5.5)-(5.6) above.

INPUT:

- `m` – positive integer defining the Fourier mode
- `a` – BH angular momentum parameter (in units of M, the BH mass)
- `r0` – Boyer-Lindquist radius of the orbit (in units of M)
- `theta` – Boyer-Lindquist colatitute θ of the observer
- `l_max` – (default: 10) upper bound in the summation over the harmonic degree ℓ in Eqs. (5.5)-(5.6)
- `algorithm_Zinf` – (default: `'spline'`) string describing the computational method for $Z^{\infty}_{\ell m}(r_0)$; allowed values are
  - `'spline'`: spline interpolation of tabulated data
  - `'1.5PN'` (only for `a=0`): 1.5-post-Newtonian expansion following E. Poisson, Phys. Rev. D **47**, 1497 (1993) [doi:10.1103/PhysRevD.47.1497]

OUTPUT:

- tuple $(\bar{A}^{\times}_m, \bar{B}^{\times}_m)$

EXAMPLES:

Let us consider the case $a = 0$ first (Schwarzschild black hole), with $m = 2$:

```
sage: from kerrgeodesic_gw import h_cross_particle_fourier
sage: a = 0
```

$h^{\times}_m$ is always zero in the direction $\theta = \pi/2$:

```
sage: h_cross_particle_fourier(2, a, 8., pi/2)  # tol 1.0e-13
(3.444996575846961e-17, 1.118234985040581e-16)
```

Let us then evaluate $h^{\times}_m$ in the direction $\theta = \pi/4$:

```
sage: h_cross_particle_fourier(2, a, 8., pi/4)  # tol 1.0e-13
(0.09841144532628172, 0.31201728756415015)
sage: h_cross_particle_fourier(2, a, 8., pi/4, l_max=5)  # tol 1.0e-13
(0.09841373523119075, 0.31202073689061305)
sage: h_cross_particle_fourier(2, a, 8., pi/4, l_max=5, algorithm_Zinf='1.5PN')  # tol 1.0e-13
(0.11744823578781578, 0.34124272645755677)
```

Values of m different from 2:

```
sage: h_cross_particle_fourier(3, a, 20., pi/4)  # tol 1.0e-13
(0.022251439699635174, -0.005354134279052387)
sage: h_cross_particle_fourier(3, a, 20., pi/4, l_max=5, algorithm_Zinf='1.5PN')  # tol 1.0e-13
(0.017782177999686274, 0.0)
sage: h_cross_particle_fourier(1, a, 8., pi/4)  # tol 1.0e-13
(0.03362589948237155, -0.008465651545641889)
sage: h_cross_particle_fourier(0, a, 8., pi/4)
(0, 0)
```

The case $a/M = 0.95$ (rapidly rotating Kerr black hole):

```
sage: a = 0.95
sage: h_cross_particle_fourier(2, a, 8., pi/4)  # tol 1.0e-13
(0.08843838892991202, 0.28159329265206867)
sage: h_cross_particle_fourier(8, a, 8., pi/4)  # tol 1.0e-13
```

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```
(-0.00014588821622195107, -8.179557811364057e-06)
sage: h_cross_particle_fourier(2, a, 2., pi/4)  # tol 1.0e-13
(-0.6021994882885746, 0.3789513303450391)
sage: h_cross_particle_fourier(8, a, 2., pi/4)  # tol 1.0e-13
(0.01045760329050054, -0.004986913120370192)
```

`kerrgeodesic_gw.gw_particle.`**`h_particle_quadrupole`**(*r0*, *u*, *theta*, *phi*, *phi0=0*, *mode='+'*)

Return the rescaled h_+ or $h_\times$ part of the gravitational radiation emitted by a particle in quasi-Newtonian circular orbit around a massive body, computed at the quadrupole approximation.

The computation is performed according to the following formulas:

$$h_+ = 2\,\frac{\mu}{r}\,\frac{M}{r_0}(1+\cos^2\theta)\cos\left[2\omega_0(t-r)+2(\phi_0-\phi)\right]$$
$$h_\times = 4\,\frac{\mu}{r}\,\frac{M}{r_0}\cos\theta\sin\left[2\omega_0(t-r)+2(\phi_0-\phi)\right]$$

where M is the mass of the central body, $\mu \ll M$ the mass of the orbiting particle, r_0 the orbital radius, $\omega_0 = \sqrt{M/r_0^3}$ the corresponding orbital angular velocity and (t, r, θ, ϕ) the coordinates of the observer.

INPUT:

- `r0` – radius of the orbit (in units of M, the mass of the central body)
- `u` – retarded time coordinate $u = t - r$ of the observer (in units of M)
- `theta` – colatitute θ of the observer
- `phi` – azimuthal coordinate ϕ of the observer
- `phi0` – (default: 0) phase factor
- `mode` – (default: `'+'`) string determining which GW polarization mode is considered; allowed values are `'+'` and `'x'`, for respectively h_+ and $h_\times$

OUTPUT:

- the rescaled waveform $(r/\mu)h$, where μ is the particle's mass, r is the radial coordinate of the observer and h is either h_+ or $h_\times$ (depending on the value of `mode`)

EXAMPLES:

Values of h_+ for $r_0 = 12M$:

```
sage: from kerrgeodesic_gw import (h_particle_quadrupole,
....:                              h_plus_particle, h_cross_particle)
sage: theta, phi = pi/3, 0
sage: r0 = 12.
sage: porb = n(2*pi*r0^1.5)  # orbital period
sage: u = porb/16
sage: h_particle_quadrupole(r0, u, theta, phi)  # tol 1.0e-13
0.1473139127471974
sage: h_plus_particle(0., r0, u, theta, phi)  # exact value, tol 1.0e-13
0.041745779012809306
sage: h_plus_particle(0., r0, u, theta, phi, l_max=5,
....:                 algorithm_Zinf='1.5PN')  # 1.5 PN approx, tol 1.0e-13
0.06396788402755646
```

Values of $h_\times$ for $r_0 = 12M$:

```
sage: h_particle_quadrupole(r0, u, theta, phi, mode='x')  # tol 1.0e-13
0.1178511301977579
sage: h_cross_particle(0., r0, u, theta, phi)  # exact value, tol 1.0e-13
0.1351232731503482
```

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```
sage: h_cross_particle(0., r0, u, theta, phi, l_max=5,
....:                  algorithm_Zinf='1.5PN')  # 1.5 PN approx, tol 1.0e-13
0.14924631043762673
```

Values of h_+ for $r_0 = 50M$:

```
sage: r0 = 50.
sage: porb = n(2*pi*r0^1.5)  # orbital period
sage: u = porb/16
sage: h_particle_quadrupole(r0, u, theta, phi)  # tol 1.0e-13
0.03535533905932738
sage: h_plus_particle(0., r0, u, theta, phi, l_max=5)  # exact value, tol 1.0e-13
0.024850367000986223
sage: h_plus_particle(0., r0, u, theta, phi, l_max=5,
....:                 algorithm_Zinf='1.5PN')  # 1.5 PN approx, tol 1.0e-13
0.026007035506911764
```

The difference between the exact value and the one resulting from the quadrupole approximation looks large, but actually this results mostly from some dephasing, as one can see on a plot:

```
sage: hp_quad = lambda t: h_particle_quadrupole(r0, t, theta, phi)
sage: hc_quad = lambda t: h_particle_quadrupole(r0, t, theta, phi, mode='x')
sage: hp = lambda t: h_plus_particle(0., r0, t, theta, phi, l_max=5)
sage: hc = lambda t: h_cross_particle(0., r0, t, theta, phi, l_max=5)
sage: umax = 2*porb
sage: plot(hp_quad, (0, umax), legend_label=r'$h_+$ quadrupole',
....:      axes_labels=[r'$(t-r_*)/M$', r'$r h/\mu$'],
....:      title=r'$r_0=50 M$, $\theta=\pi/3$',
....:      gridlines=True, frame=True, axes=False) + \
....: plot(hp, (0, umax), color='red', legend_label=r'$h_+$ exact') + \
....: plot(hc_quad, (0, umax), linestyle='--',
....:      legend_label=r'$h_\times$ quadrupole') + \
....: plot(hc, (0, umax), color='red', linestyle='--',
....:      legend_label=r'$h_\times$ exact')
Graphics object consisting of 4 graphics primitives
```

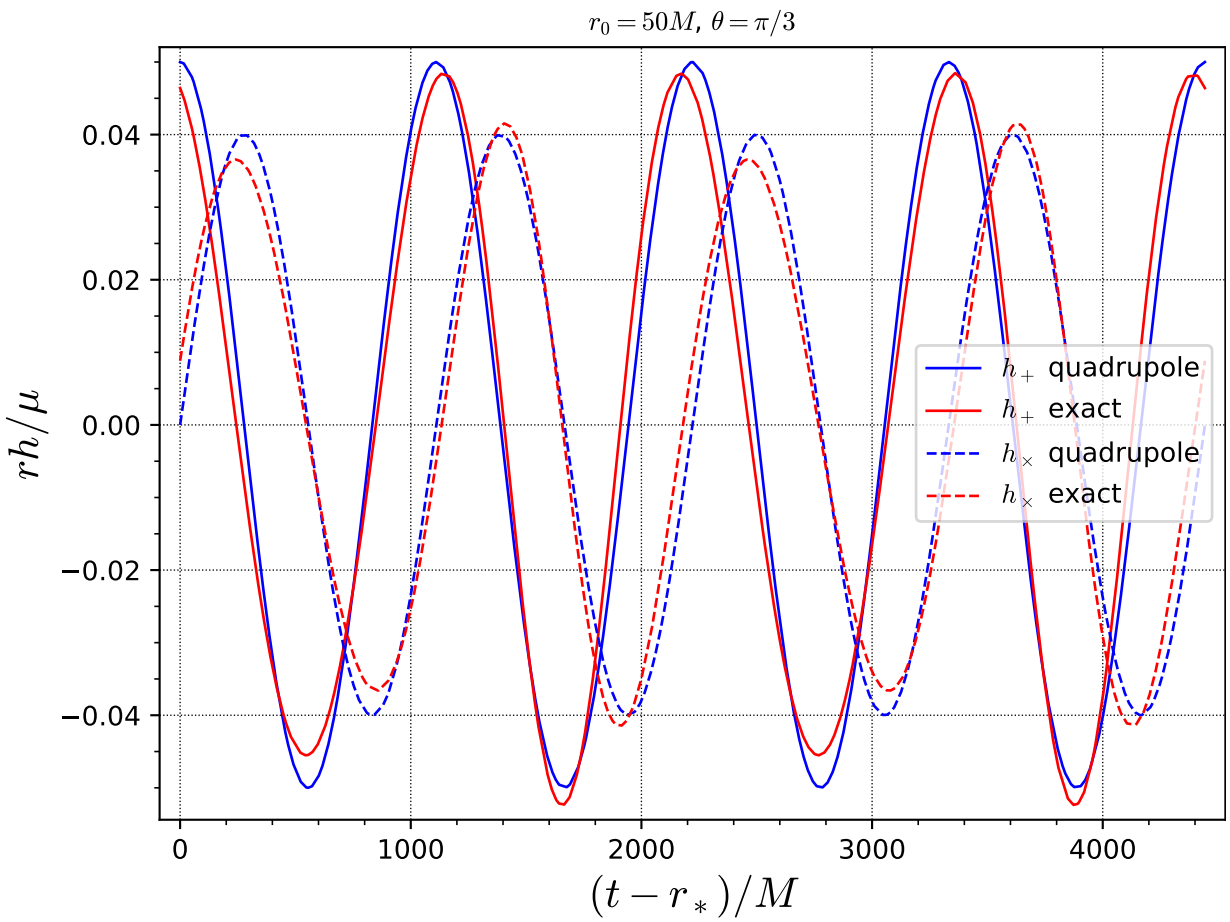
`kerrgeodesic_gw.gw_particle.`**`h_particle_signal`**(*a*, *r0*, *theta*, *phi*, *u_min*, *u_max*, *mode='+'*, *nb_points=100*, *phi0=0*, *l_max=10*, *m_min=1*, *approximation=None*, *store=None*)

Return a time sequence of the h_+ or the $h_\times$ part of the gravitational radiation from a particle in circular orbit around a Kerr black hole.

Note: It is more efficient to use this function than to perform a loop over *h_plus_particle()* or *h_cross_particle()*. Indeed, the Fourier modes, which involve the computation of spin-weighted spheroidal harmonics and of the functions $Z^\infty_{\ell m}(r_0)$, are evaluated once for all, prior to the loop on the retarded time u.

INPUT:

- `a` – BH angular momentum parameter (in units of M)
- `r0` – Boyer-Lindquist radius of the orbit (in units of M)
- `theta` – Boyer-Lindquist colatitute θ of the observer
- `phi` – Boyer-Lindquist azimuthal coordinate ϕ of the observer
- `u_min` – lower bound of the retarded time coordinate of the observer (in units of the black hole mass M): $u = t - r_*$, where t is the Boyer-Lindquist time coordinate and r_* is the tortoise coordinate
- `u_max` – upper bound of the retarded time coordinate of the observer (in units of the black hole mass M)


$r_0 = 50M$, $\theta = \pi/3$
0.04
0.02
0.00
−0.02
−0.04
rh/μ
0
1000
2000
3000
4000
$(t - r_*)/M$
h_+ quadrupole
h_+ exact
$h_\times$ quadrupole
$h_\times$ exact

- `mode` – (default: `'+'`) string determining which GW polarization mode is considered; allowed values are `'+'` and `'x'`, for respectively h_+ and $h_\times$
- `nb_points` – (default: 100) number of points in the interval `(u_min, u_max)`
- `phi0` – (default: 0) phase factor
- `l_max` – (default: 10) upper bound in the summation over the harmonic degree ℓ
- `m_min` – (default: 1) lower bound in the summation over the Fourier mode m
- `approximation` – (default: `None`) string describing the computational method; allowed values are
  - `None`: exact computation
  - `'1.5PN'` (only for `a=0`): 1.5-post-Newtonian expansion following E. Poisson, Phys. Rev. D **47**, 1497 (1993) [doi:10.1103/PhysRevD.47.1497]
  - `'quadrupole'` (only for `a=0`): quadrupole approximation (0-post-Newtonian); see *`h_particle_quadrupole()`*
- `store` – (default: `None`) string containing a file name for storing the time sequence; if `None`, no storage is attempted

OUTPUT:

- a list of `nb_points` pairs $(u, rh/\mu)$, where u is the retarded time, h is either h_+ or $h_\times$ depending on the parameter `mode`, μ is the particle mass and r is the Boyer-Lindquist radial coordinate of the observer

EXAMPLES:

h_+ signal at $\theta = \pi/2$ from a particle at the ISCO of a Schwarzschild black hole ($a = 0$, $r_0 = 6M$):

```
sage: from kerrgeodesic_gw import h_particle_signal
sage: h_particle_signal(0., 6., pi/2, 0., 0., 200., nb_points=9)  # tol 1.0e-13
[(0.000000000000000, 0.1536656546005028),
 (25.0000000000000, -0.2725878162016865),
 (50.0000000000000, 0.3525164756465054),
 (75.0000000000000, 0.047367530900643974),
 (100.000000000000, -0.06816472285771447),
 (125.000000000000, -0.10904082076122341),
 (150.000000000000, 0.11251491162759894),
 (175.000000000000, 0.2819301792449237),
 (200.000000000000, -0.24646401049292863)]
```

Storing the sequence in a file:

```
sage: h = h_particle_signal(0., 6., pi/2, 0., 0., 200.,
....:                       nb_points=9, store='h_plus.d')
```

The $h_\times$ signal, for $\theta = \pi/4$:

```
sage: h_particle_signal(0., 6., pi/4, 0., 0., 200., nb_points=9, mode='x')  # tol 1.0e-13
[(0.000000000000000, 0.275027796440582),
 (25.0000000000000, -0.18713017721920192),
 (50.0000000000000, 0.2133141583155321),
 (75.0000000000000, -0.531073507307601),
 (100.000000000000, 0.3968872953624949),
 (125.000000000000, -0.4154274307718398),
 (150.000000000000, 0.5790969355083798),
 (175.000000000000, -0.24074783639714234),
 (200.000000000000, 0.22869838143661578)]
```

`kerrgeodesic_gw.gw_particle.h_plus_particle`(*a*, *r0*, *u*, *theta*, *phi*, *phi0=0*, *l_max=10*, *m_min=1*, *algorithm_Zinf='spline'*)

Return the rescaled h_+-part of the gravitational radiation emitted by a particle in circular orbit around a Kerr black hole.

The computation is based on Eq. (5.2) above.

INPUT:

- `a` – BH angular momentum parameter (in units of M, the BH mass)
- `r0` – Boyer-Lindquist radius of the orbit (in units of M)
- `u` – retarded time coordinate of the observer (in units of M): $u = t - r_*$, where t is the Boyer-Lindquist time coordinate and r_* is the tortoise coordinate
- `theta` – Boyer-Lindquist colatitute θ of the observer
- `phi` – Boyer-Lindquist azimuthal coordinate ϕ of the observer
- `phi0` – (default: 0) phase factor
- `l_max` – (default: 10) upper bound in the summation over the harmonic degree ℓ
- `m_min` – (default: 1) lower bound in the summation over the Fourier mode m
- `algorithm_Zinf` – (default: `'spline'`) string describing the computational method for $Z^\infty_{\ell m}(r_0)$; allowed values are
  - `'spline'`: spline interpolation of tabulated data
  - `'1.5PN'` (only for `a=0`): 1.5-post-Newtonian expansion following E. Poisson, Phys. Rev. D **47**, 1497 (1993) [doi:10.1103/PhysRevD.47.1497]

OUTPUT:

- the rescaled waveform $(r/\mu)h_+$, where μ is the particle's mass and r is the Boyer-Lindquist radial coordinate of the observer

EXAMPLES:

Let us consider the case $a = 0$ (Schwarzschild black hole) and $r_0 = 6M$ (emission from the ISCO):

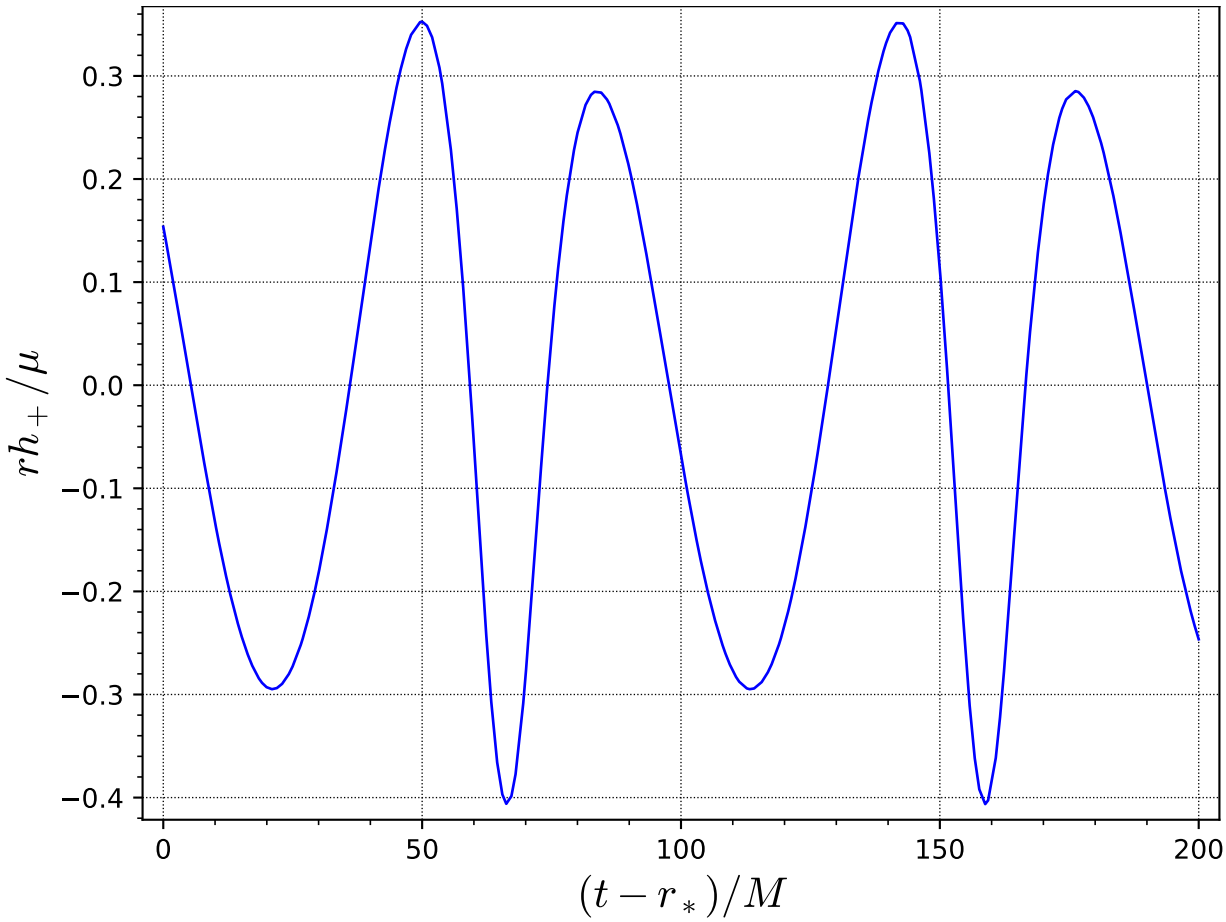
```
sage: from kerrgeodesic_gw import h_plus_particle
sage: a = 0
sage: h_plus_particle(a, 6., 0., pi/2, 0.)  # tol 1.0e-13
0.1536656546005028
sage: h_plus_particle(a, 6., 0., pi/2, 0., l_max=5)  # tol 1.0e-13
0.157759938177291
sage: h_plus_particle(a, 6., 0., pi/2, 0., l_max=5, algorithm_Zinf='1.5PN')  # tol 1.0e-13
0.22583887001798497
```

For an orbit of larger radius ($r_0 = 12M$), the 1.5-post-Newtonian approximation is in better agreement with the exact computation:

```
sage: h_plus_particle(a, 12., 0., pi/2, 0.)  # tol 1.0e-13
0.11031251832047866
sage: h_plus_particle(a, 12., 0., pi/2, 0., l_max=5, algorithm_Zinf='1.5PN')  # tol 1.0e-13
0.12935832450325302
```

A plot of the waveform generated by a particle orbiting at the ISCO:

```
sage: hp = lambda u: h_plus_particle(a, 6., u, pi/2, 0.)
sage: plot(hp, (0, 200.), axes_labels=[r'$(t-r_*)/M$', r'$r h_+/\mu$'],
....:      gridlines=True, frame=True, axes=False)
Graphics object consisting of 1 graphics primitive
```


0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
−0.4
rh_+/μ
0
50
100
150
200
$(t - r_*)/M$

Case $a/M = 0.95$ (rapidly rotating Kerr black hole):

```
sage: a = 0.95
sage: h_plus_particle(a, 2., 0., pi/2, 0.)  # tol 1.0e-13
0.20326150400852214
```

Assessing the importance of the mode $m = 1$:

```
sage: h_plus_particle(a, 2., 0., pi/2, 0., m_min=2)  # tol 1.0e-13
0.21845811047370495
```

`kerrgeodesic_gw.gw_particle.`**`h_plus_particle_fourier`**(*m*, *a*, *r0*, *theta*, *l_max=10*, *algorithm_Zinf='spline'*)

Return the Fourier mode of a given order m of the rescaled h_+-part of the gravitational wave emitted by a particle in circular orbit around a Kerr black hole.

The rescaled Fourier mode of order m received at the location (t, r, θ, ϕ) is

$$\frac{r}{\mu} h_m^+ = \bar{A}_m^+ \cos(m\psi) + \bar{B}_m^+ \sin(m\psi)$$

where μ is the particle mass and $\psi := \omega_0(t - r_*) - \phi$, ω_0 being the orbital frequency of the particle and r_* the tortoise coordinate corresponding to r and $\bar{A}_m^+$ and $\bar{B}_m^+$ are given by Eqs. (5.3)-(5.4) above.

INPUT:

- `m` – positive integer defining the Fourier mode
- `a` – BH angular momentum parameter (in units of M, the BH mass)
- `r0` – Boyer-Lindquist radius of the orbit (in units of M)
- `theta` – Boyer-Lindquist colatitute θ of the observer
- `l_max` – (default: 10) upper bound in the summation over the harmonic degree ℓ in Eqs. (5.3)-(5.4)
- `algorithm_Zinf` – (default: `'spline'`) string describing the computational method for $Z_{\ell m}^{\infty}(r_0)$; allowed values are
  - `'spline'`: spline interpolation of tabulated data
  - `'1.5PN'` (only for `a=0`): 1.5-post-Newtonian expansion following E. Poisson, Phys. Rev. D **47**, 1497 (1993) [doi:10.1103/PhysRevD.47.1497]

OUTPUT:

- tuple $(\bar{A}_m^+, \bar{B}_m^+)$

EXAMPLES:

Let us consider the case $a = 0$ first (Schwarzschild black hole), with $m = 2$:

```
sage: from kerrgeodesic_gw import h_plus_particle_fourier
sage: a = 0
sage: h_plus_particle_fourier(2, a, 8., pi/2)  # tol 1.0e-13
(0.2014580652208302, -0.06049343736886148)
sage: h_plus_particle_fourier(2, a, 8., pi/2, l_max=5)  # tol 1.0e-13
(0.20146097329552273, -0.060495372034569186)
sage: h_plus_particle_fourier(2, a, 8., pi/2, l_max=5, algorithm_Zinf='1.5PN')  # tol 1.0e-13
(0.2204617125753912, -0.0830484439639611)
```

Values of m different from 2:

```
sage: h_plus_particle_fourier(3, a, 20., pi/2)  # tol 1.0e-13
(-0.005101595598729037, -0.021302121442654077)
sage: h_plus_particle_fourier(3, a, 20., pi/2, l_max=5, algorithm_Zinf='1.5PN')  # tol 1.0e-13
(0.0, -0.016720919174427588)
sage: h_plus_particle_fourier(1, a, 8., pi/2)  # tol 1.0e-13
(-0.014348477201223874, -0.05844679575244101)
sage: h_plus_particle_fourier(0, a, 8., pi/2)
(0, 0)
```

The case $a/M = 0.95$ (rapidly rotating Kerr black hole):

```
sage: a = 0.95
sage: h_plus_particle_fourier(2, a, 8., pi/2)  # tol 1.0e-13
(0.182748773431646, -0.05615306925896938)
sage: h_plus_particle_fourier(8, a, 8., pi/2)  # tol 1.0e-13
(-4.724709221209198e-05, 0.0006867183495228116)
sage: h_plus_particle_fourier(2, a, 2., pi/2)  # tol 1.0e-13
(0.170040287761701 4, 0.33693580916655747)
sage: h_plus_particle_fourier(8, a, 2., pi/2)  # tol 1.0e-13
(-0.009367442995129153, -0.03555092085651877)
```

kerrgeodesic_gw.gw_particle.**plot_h_particle**(*a*, *r0*, *theta*, *phi*, *u_min*, *u_max*, *plot_points=200*, *phi0=0*, *l_max=10*, *m_min=1*, *approximation=None*, *mode=('+', 'x')*, *color=None*, *linestyle=None*, *legend_label=('h_+', '$h_\\\\times$')*, *xlabel='$(t - r_*)/M$'*, *ylabel=None*, *title=None*, *gridlines=True*)

Plot the gravitational waveform emitted by a particle in circular orbit around a Kerr black hole.

INPUT:

- `a` – BH angular momentum parameter (in units of M, the black hole mass)
- `r0` – Boyer-Lindquist radius of the orbit (in units of M)
- `theta` – Boyer-Lindquist colatitute θ of the observer
- `phi` – Boyer-Lindquist azimuthal coordinate ϕ of the observer
- `u_min` – lower bound of the retarded time coordinate of the observer (in units of M): $u = t - r_*$, where t is the Boyer-Lindquist time coordinate and r_* is the tortoise coordinate
- `u_max` – upper bound of the retarded time coordinate of the observer (in units of M)
- `plot_points` – (default: 200) number of points involved in the sampling of the interval `(u_min, u_max)`
- `phi0` – (default: 0) phase factor
- `l_max` – (default: 10) upper bound in the summation over the harmonic degree ℓ
- `m_min` – (default: 1) lower bound in the summation over the Fourier mode m
- `approximation` – (default: `None`) string describing the computational method; allowed values are
  - `None`: exact computation
  - `'1.5PN'` (only for `a=0`): 1.5-post-Newtonian expansion following E. Poisson, Phys. Rev. D **47**, 1497 (1993) [doi:10.1103/PhysRevD.47.1497]
  - `'quadrupole'` (only for `a=0`): quadrupole approximation (0-post-Newtonian); see *`h_particle_quadrupole()`*
- `mode` – (default: `('+', 'x')`) string determining the plotted quantities: allowed values are `'+'` and `'x'`, for respectively h_+ and $h_\times$, as well as `('+', 'x')` for plotting both polarization modes
- `color` – (default: `None`) a color (if `mode = '+'` or `'x'`) or a pair of colors (if `mode = ('+', 'x')`) for the plot(s); if `None`, the default colors are `'blue'` for h_+ and `'red'` for $h_\times$

- `linestyle` – (default: `None`) a line style (if `mode = '+'` or `'x'`) or a pair of line styles (if `mode = ('+', 'x')`) for the plot(s); if `None`, the default style is a solid line
- `legend_label` – (default: `(r'$h_+$', r'$h_\times$')`) labels for the plots of h_+ and $h_\times$; used only if `mode` is `('+', 'x')`
- `xlabel` – (default: `r'$(t - r_*)/M$'`) label of the x-axis
- `ylabel` – (default: `None`) label of the y-axis; if `None`, `r'$(r/\mu)\, h_+$'` is used for `mode = '+'`, `r'$(r/\mu)\, h_\times$'` for `mode = 'x'` and `r'$(r/\mu)\, h_{+,\times}$'` for `mode = ('+', 'x')`
- `title` – (default: `None`) plot title; if `None`, the title is generated from `a`, `r0` and `theta` (see the example below)
- `gridlines` – (default: `True`) indicates whether the gridlines are to be plotted

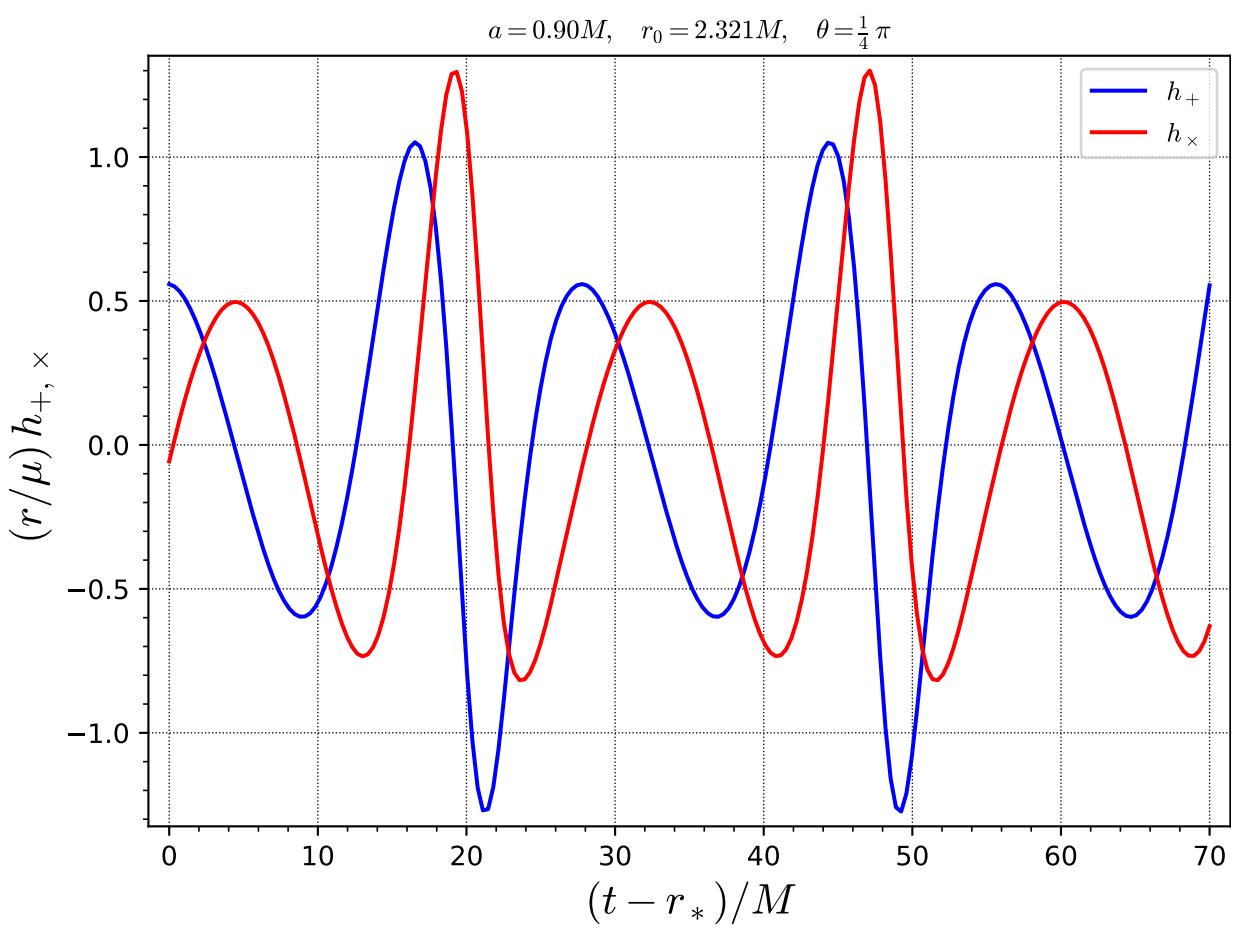
OUTPUT:

- a graphics object

EXAMPLES:

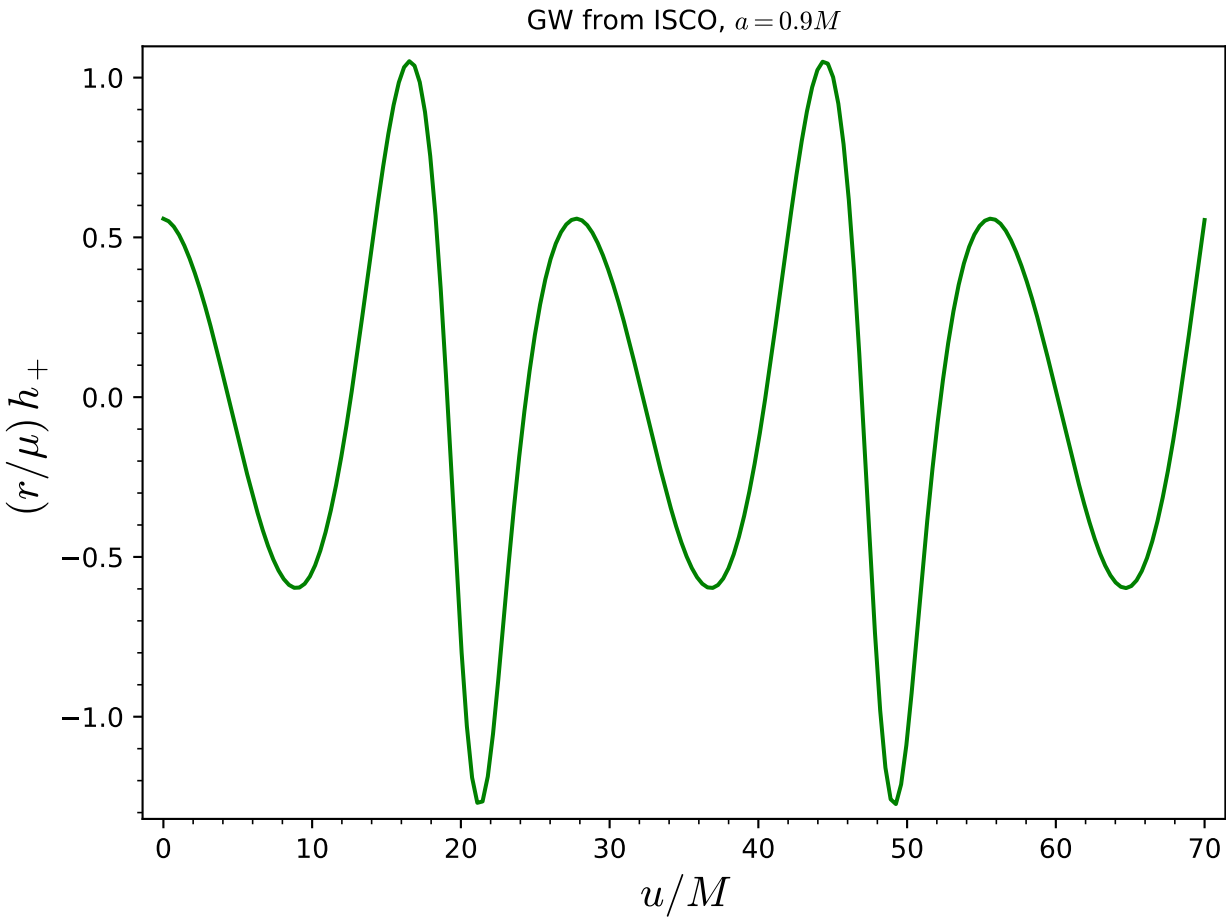
Plot of the gravitational waveform generated by a particle orbiting at the ISCO of a Kerr black hole with $a = 0.9M$:

```
sage: from kerrgeodesic_gw import plot_h_particle
sage: plot_h_particle(0.9, 2.321, pi/4, 0., 0., 70.)
Graphics object consisting of 2 graphics primitives
```



Plot of h_+ only, with some non-default options:

```
sage: plot_h_particle(0.9, 2.321, pi/4, 0., 0., 70., mode='+',
....:                 color='green', xlabel=r'$u/M$', gridlines=False,
....:                 title='GW from ISCO, $a=0.9M$')
Graphics object consisting of 1 graphics primitive
```



kerrgeodesic_gw.gw_particle.**plot_spectrum_particle**(*a*, *r0*, *theta*, *mode='+'*, *m_max=10*, *l_max=10*, *algorithm_Zinf='spline'*, *color='blue'*, *linestyle='-'*, *thickness=2*, *legend_label=None*, *offset=0*, *xlabel=None*, *ylabel=None*, *title=None*, *gridlines=True*)

Plot the Fourier spectrum of the gravitational radiation emitted by a particle in equatorial circular orbit around a Kerr black hole.

The *Fourier spectrum* is defined by the following sequence indexed by m:

$$H_m^{+,\times}(\theta) := \frac{r}{\mu}\sqrt{(A_m^{+,\times}(\theta))^2 + (B_m^{+,\times}(\theta))^2}$$

where the functions A_m^+, B_m^+, $A_m^\times$, $B_m^\times$ are defined by (5.2).

INPUT:

- a – BH angular momentum parameter (in units of M, the BH mass)
- r0 – Boyer-Lindquist radius of the orbit (in units of M)

- `theta` – Boyer-Lindquist colatitute θ of the observer
- `mode` – (default: `'+'`) string determining which GW polarization mode is considered; allowed values are `'+'` and `'x'`, for respectively h_+ and $h_\times$
- `m_max` – (default: 10) maximal value of m
- `l_max` – (default: 10) upper bound in the summation over the harmonic degree ℓ
- `algorithm_Zinf` – (default: `'spline'`) string describing the computational method for $Z^\infty_{\ell m}(r_0)$; allowed values are
  - `'spline'`: spline interpolation of tabulated data
  - `'1.5PN'` (only for `a=0`): 1.5-post-Newtonian expansion following E. Poisson, Phys. Rev. D **47**, 1497 (1993) [doi:10.1103/PhysRevD.47.1497]
- `color` – (default: `'blue'`) color of vertical lines
- `linestyle` – (default: `'-'`) style of vertical lines
- `legend_label` – (default: `None`) legend label for this spectrum
- `offset` – (default: 0) horizontal offset for the position of the vertical lines
- `xlabel` – (default: `None`) label of the x-axis; if none is provided, the label is set to m
- `ylabel` – (default: `None`) label of the y-axis; if none is provided, the label is set to $H_m^{+,\times}$
- `title` – (default: `None`) plot title; if `None`, the title is generated from `a`, `r0` and `theta`
- `gridlines` – (default: `True`) indicates whether the gridlines are to be plotted

OUTPUT:

- a graphics object

EXAMPLES:

Spectrum of gravitational radiation generated by a particle orbiting at the ISCO of a Schwarzschild black hole ($a = 0$, $r_0 = 6M$):

```
sage: from kerrgeodesic_gw import plot_spectrum_particle
sage: plot_spectrum_particle(0, 6., pi/2)
Graphics object consisting of 10 graphics primitives
```

`kerrgeodesic_gw.gw_particle.`**`radiated_power_particle`**(*a*, *r0*, *l_max=None*, *m_min=1*, *approximation=None*)

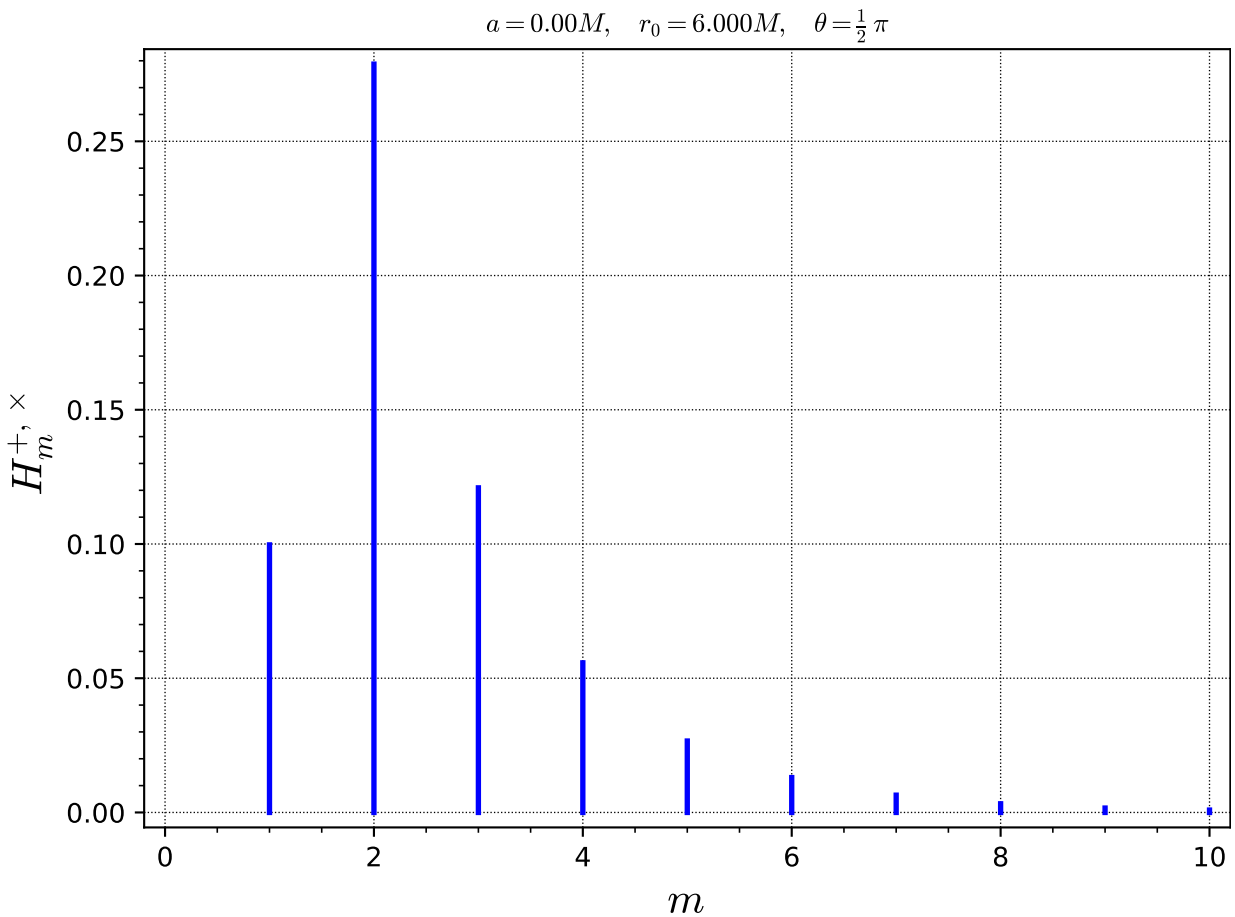
Return the total (i.e. summed over all directions) power of the gravitational radiation emitted by a particle in circular orbit around a Kerr black hole.

The total radiated power (or *luminosity*) L is computed according to the formula:

$$L = \frac{\mu^2}{4\pi} \sum_{\ell=2}^{\infty} \sum_{\substack{m=-\ell \\ m\neq 0}}^{\ell} \frac{|Z^\infty_{\ell m}(r_0)|^2}{(m\omega_0)^2}$$

INPUT:

- `a` – BH angular momentum parameter (in units of M, the BH mass)
- `r0` – Boyer-Lindquist radius of the orbit (in units of M)
- `l_max` – (default: `None`) upper bound in the summation over the harmonic degree ℓ; if `None`, `l_max` is determined automatically from the available tabulated data
- `m_min` – (default: 1) lower bound in the summation over the Fourier mode m


$a = 0.00M, \quad r_0 = 6.000M, \quad \theta = \frac{1}{2}\pi$
0.25
0.20
0.15
0.10
0.05
0.00
$H_m^{+,\times}$
0
2
4
6
8
10
m

- `approximation` – (default: `None`) string describing the computational method; allowed values are
  - `None`: exact computation
  - `'1.5PN'` (only for `a=0`): 1.5-post-Newtonian expansion following E. Poisson, Phys. Rev. D **47**, 1497 (1993) [doi:10.1103/PhysRevD.47.1497]
  - `'quadrupole'` (only for `a=0`): quadrupole approximation (0-post-Newtonian); see *h_particle_quadrupole()*

OUTPUT:

- rescaled radiated power $L\,(M/\mu)^2$, where M is the BH mass and μ the mass of the orbiting particle.

EXAMPLES:

Power radiated by a particle at the ISCO of a Schwarzschild BH:

```
sage: from kerrgeodesic_gw import radiated_power_particle
sage: radiated_power_particle(0, 6.)
0.000937262782177525
```

Power radiated by a particle at the ISCO of a rapidly rotating Kerr BH ($a = 0.98M$):

```
sage: radiated_power_particle(0.98, 1.61403)  # tol 1.0e-13
0.08629927053494096
```

Power computed according to various approximations:

```
sage: radiated_power_particle(0, 6., approximation='1.5PN')  # tol 1.0e-13
0.0010435864384751143
sage: radiated_power_particle(0, 6., approximation='quadrupole')  # tol 1.0e-13
0.000823045267489712
```

Let us check that at large radius, these approximations get closer to the actual result:

```
sage: radiated_power_particle(0, 50.)  # tol 1.0e-13
1.9624555021692417e-08
sage: radiated_power_particle(0, 50., approximation='1.5PN')  # tol 1.0e-13
1.958318200436188e-08
sage: radiated_power_particle(0, 50., approximation='quadrupole')  # tol 1.0e-13
2.0480000000000002e-08
```

`kerrgeodesic_gw.gw_particle.`**`secular_frequency_change`**(*a*, *r0*, *l_max=None*, *m_min=1*, *approximation=None*)

Return the gravitational-radiation induced change of the orbital frequency of a particle in circular orbit around a Kerr black hole.

INPUT:

- `a` – BH angular momentum parameter (in units of M, the BH mass)
- `r0` – Boyer-Lindquist radius of the orbit (in units of M)
- `l_max` – (default: `None`) upper bound in the summation over the harmonic degree ℓ; if `None`, `l_max` is determined automatically from the available tabulated data
- `m_min` – (default: 1) lower bound in the summation over the Fourier mode m
- `approximation` – (default: `None`) string describing the computational method; allowed values are
  - `None`: exact computation
  - `'1.5PN'` (only for `a=0`): 1.5-post-Newtonian expansion following E. Poisson, Phys. Rev. D **47**, 1497 (1993) [doi:10.1103/PhysRevD.47.1497]

- `'quadrupole'` (only for `a=0`): quadrupole approximation (0-post-Newtonian); see *h_particle_quadrupole()*

OUTPUT:

- rescaled fractional change in orbital frequency $\dot{\omega}_0/\omega_0\,(M^2/\mu)$, where M is the BH mass and μ the mass of the orbiting particle.

EXAMPLES:

Relative change in orbital frequency change at $r_0 = 10M$ around a Schwarzschild black hole:

```
sage: from kerrgeodesic_gw import secular_frequency_change
sage: secular_frequency_change(0, 10)  # tol 1.0e-13
0.002701529506901975
sage: secular_frequency_change(0, 10, approximation='quadrupole')  # tol 1.0e-13
0.00192
```

At larger distance, the quadrupole approximation works better:

```
sage: secular_frequency_change(0, 50)  # tol 1.0e-13
3.048598175592674e-06
sage: secular_frequency_change(0, 50, approximation='quadrupole')  # tol 1.0e-13
3.072e-06
```

At the ISCO, $\dot{\omega}_0$ diverges:

```
sage: secular_frequency_change(0, 6)
+infinity
```

while the quadrupole approximation would have predict a finite value there:

```
sage: secular_frequency_change(0, 6, approximation='quadrupole')  # tol 1.0e-13
0.014814814814814817
```

Case of a Kerr black hole:

```
sage: secular_frequency_change(0.5, 6)  # tol 1.0e-13
0.020393023677356764
```

CHAPTER

SIX

GRAVITATIONAL RADIATION BY A BLOB OF MATTER

Gravitational radiation by a blob of matter on a circular orbit around a Kerr black hole

This module implements the following functions:

- *h_blob()*: evaluates rh_+ or $rh_\times$
- *h_blob_signal()*: time sequence of rh_+/μ or $rh_\times/\mu$
- *h_toy_model_semi_analytic()*: rh_+/μ and $rh_\times/\mu$ from a semi-analytic approximation based on a constant density blob
- *surface_density_toy_model()*: $\Sigma(\bar{r},\bar{\phi})$ as an indicator function
- *surface_density_gaussian()*: $\Sigma(\bar{r},\bar{\phi})$ as a Gaussian profile
- *blob_mass()*: mass of the blob of matter, by integration of $\Sigma(\bar{r},\bar{\phi})$

`kerrgeodesic_gw.gw_blob.`**`blob_mass`**`(`*a*, *surf_dens*, *param_surf_dens*, *integ_range*, *epsabs=1e-08*, *epsrel=1e-08*`)`

Compute the mass of a blob of matter by integrating its surface density.

INPUT:

- `a` – BH angular momentum parameter (in units of M)
- `surf_dens` – surface density function Σ; must take three arguments: `(r_bar, phi_bar, param_surf_dens)`, where
  - `r_bar` is the Boyer-Lindquist radial coordinate $\bar{r}$ in the matter blob
  - `phi_bar` is the Boyer-Lindquist azimuthal coordinate $\bar{\phi}$ in the matter blob
  - `param_surf_dens` are parameters defining the function $\Sigma(\bar{r},\bar{\phi})$
- `param_surf_dens` – parameters to be passed as the third argument to the function `surf_dens` (see above)
- `integ_range` – tuple $(\bar{r}_1,\bar{r}_2,\bar{\phi}_1,\bar{\phi}_2)$ defining the integration range
- `epsabs` – (default: 1e-8) absolute tolerance of the inner 1-D quadrature integration
- `epsrel` – (default: 1e-8) relative tolerance of the inner 1-D integrals

OUTPUT:

- a pair `(mu, err)`, where `mu` is the mass of the matter blob and `err` is an estimate of the absolute error in the computation of the integral

EXAMPLES:

```
sage: from kerrgeodesic_gw import blob_mass, surface_density_gaussian
sage: param = [6.5, 0., 0.3]
sage: integ_range = [6, 7, -0.1, 0.1]
sage: blob_mass(0., surface_density_gaussian, param, integ_range)  # tol 1.0e-13
(0.3328779622200767, 5.671806593829388e-10)
sage: blob_mass(0., surface_density_gaussian, param, integ_range,  # tol 1.0e-13
....:           epsabs=1.e-3)
(0.33287796222007715, 1.109366597806814e-06)
```

kerrgeodesic_gw.gw_blob.**h_blob**(*u*, *theta*, *phi*, *a*, *surf_dens*, *param_surf_dens*, *integ_range*, *mode='+'*, *l_max=10*, *m_min=1*, *epsabs=1e-06*, *epsrel=1e-06*)

Return the rescaled value of h_+ or $h_\times$ (depending on the parameter `mode`) for the gravitational radiation emitted by a matter blob in circular orbit around a Kerr black hole.

INPUT:

- `u` – retarded time coordinate of the observer (in units of M, the BH mass): $u = t - r_*$, where t is the Boyer-Lindquist time coordinate and r_* is the tortoise coordinate
- `theta` – Boyer-Lindquist colatitute θ of the observer
- `phi` – Boyer-Lindquist azimuthal coordinate ϕ of the observer
- `a` – BH angular momentum parameter (in units of M)
- `surf_dens` – surface density function Σ; must take three arguments: `(r_bar, phi_bar, param_surf_dens)`, where
  - `r_bar` is the Boyer-Lindquist radial coordinate $\bar{r}$ in the matter blob
  - `phi_bar` is the Boyer-Lindquist azimuthal coordinate $\bar{\phi}$ in the matter blob
  - `param_surf_dens` are parameters defining the function $\Sigma(\bar{r}, \bar{\phi})$
- `param_surf_dens` – parameters to be passed as the third argument to the function `surf_dens` (see above)
- `integ_range` – tuple $(\bar{r}_1, \bar{r}_2, \bar{\phi}_1, \bar{\phi}_2)$ defining the integration range
- `mode` – (default: `'+'`) string determining which GW polarization mode is considered; allowed values are `'+'` and `'x'`, for respectively h_+ and $h_\times$
- `l_max` – (default: 10) upper bound in the summation over the harmonic degree ℓ
- `m_min` – (default: 1) lower bound in the summation over the Fourier mode m
- `epsabs` – (default: 1e-6) absolute tolerance passed directly to the inner 1-D quadrature integration
- `epsrel` – (default: 1e-6) relative tolerance of the inner 1-D integrals

OUTPUT:

- a pair `(rh, err)`, where `rh` is the rescaled waveform rh_+ (resp. $rh_\times$) if `mode = '+'` (resp. `'x'`), r being the Boyer-Lindquist radial coordinate of the observer, and `err` is an estimate of the absolute error in the computation of the integral

EXAMPLES:

Gravitational emission h_+ from a constant density blob close to the ISCO of a Schwarzschild black hole:

```
sage: from kerrgeodesic_gw import h_blob, surface_density_toy_model
sage: a = 0
sage: param_surf_dens = [6.5, 0, 0.6, 0.1]
sage: integ_range = [6.3, 6.7, -0.04, 0.04]
sage: h_blob(0., pi/2, 0., a, surface_density_toy_model,  # tol 1.0e-13
....:        param_surf_dens, integ_range)
(0.03688373245628765, 9.872900827148663e-10)
```

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```
sage: h_blob(0., pi/2, 0., a, surface_density_toy_model,  # tol 1.0e-13
....:        param_surf_dens, integ_range, l_max=5)
(0.037532424224875585, 1.3788099591183387e-09)
```

The $h_\times$ part at $\theta = \pi/4$:

```
sage: h_blob(0., pi/4, 0., a, surface_density_toy_model,  # tol 1.0e-13
....:        param_surf_dens, integ_range, mode='x')
(0.06203815455135455, 1.972915290337975e-09)
sage: h_blob(0., pi/4, 0., a, surface_density_toy_model,  # tol 1.0e-13
....:        param_surf_dens, integ_range, mode='x', l_max=5)
(0.06121422594295032, 1.924590678064715e-09)
```

kerrgeodesic_gw.gw_blob.**h_blob_signal**(*u_min*, *u_max*, *theta*, *phi*, *a*, *surf_dens*, *param_surf_dens*, *integ_range*, *mode='+'*, *nb_points=100*, *l_max=10*, *m_min=1*, *epsabs=1e-06*, *epsrel=1e-06*, *store=None*, *verbose=True*)

Return a time sequence of h_+ or $h_\times$ (depending on the parameter `mode`) describing the gravitational radiation from a matter blob orbiting a Kerr black hole.

INPUT:

- `u_min` – lower bound of the retarded time coordinate of the observer (in units of M, the BH mass): $u = t - r_*$, where t is the Boyer-Lindquist time coordinate and r_* is the tortoise coordinate
- `u_max` – upper bound of the retarded time coordinate of the observer (in units of M)
- `theta` – Boyer-Lindquist colatitute θ of the observer
- `phi` – Boyer-Lindquist azimuthal coordinate ϕ of the observer
- `a` – BH angular momentum parameter (in units of M)
- `surf_dens` – surface density function Σ; must take three arguments: `(r_bar, phi_bar, param_surf_dens)`, where
  - `r_bar` is the Boyer-Lindquist radial coordinate $\bar{r}$ in the matter blob
  - `phi_bar` is the Boyer-Lindquist azimuthal coordinate $\bar{\phi}$ in the matter blob
  - `param_surf_dens` are parameters defining the function $\Sigma(\bar{r}, \bar{\phi})$
- `param_surf_dens` – parameters to be passed as the third argument to the function `surf_dens` (see above)
- `integ_range` – tuple $(\bar{r}_1, \bar{r}_2, \bar{\phi}_1, \bar{\phi}_2)$ defining the integration range
- `mode` – (default: `'+'`) string determining which GW polarization mode is considered; allowed values are `'+'` and `'x'`, for respectively h_+ and $h_\times$
- `nb_points` – (default: 100) number of points in the interval `(u_min, u_max)`
- `l_max` – (default: 10) upper bound in the summation over the harmonic degree ℓ
- `m_min` – (default: 1) lower bound in the summation over the Fourier mode m
- `epsabs` – (default: 1e-6) absolute tolerance passed directly to the inner 1-D quadrature integration
- `epsrel` – (default: 1e-6) relative tolerance of the inner 1-D integrals
- `store` – (default: `None`) string containing a file name for storing the time sequence; if `None`, no storage is attempted
- `verbose` – (default: `True`) boolean determining whether to monitor the progress of the sequence computation

OUTPUT:

- a list of `nb_points` pairs $(u, rh_+/\mu)$ or $(u, rh_\times/\mu)$ (depending on `mode`), where μ is the blob's mass and r is the Boyer-Lindquist radial coordinate of the observer

EXAMPLES:

h_+ sequence for a Gaussian-density matter blob around a Schwarzschild black hole:

```
sage: from kerrgeodesic_gw import h_blob_signal, surface_density_gaussian
sage: param_surf_dens = [6.5, 0., 0.3]
sage: integ_range = [6, 7, -0.1, 0.1]
sage: a = 0
sage: h_blob_signal(0., 100., pi/4, 0., a, surface_density_gaussian,  # tol 1.0e-13
....:               param_surf_dens, integ_range, nb_points=5,
....:               epsrel=1e-4, verbose=False)
[(0.000000000000000, 0.2988585811681569),
 (25.0000000000000, -0.3902717353505495),
 (50.0000000000000, 0.4196374371978159),
 (75.0000000000000, -0.4729306367862637),
 (100.000000000000, 0.3464193312741741)]
```

The corresponding $h_\times$ sequence:

```
sage: h_blob_signal(0., 100., pi/4, 0., a, surface_density_gaussian,  # tol 1.0e-13
....:               param_surf_dens, integ_range, mode='x',
....:               nb_points=5, epsrel=1e-4, verbose=False)
[(0.000000000000000, 0.2481044664324047 2),
 (25.0000000000000, -0.060862007486422516),
 (50.0000000000000, -0.12309577666126706),
 (75.0000000000000, -0.0017707729883368948),
 (100.000000000000, 0.08100026787377135)]
```

`kerrgeodesic_gw.gw_blob.`**`h_toy_model_semi_analytic`**(*u*, *theta*, *phi*, *a*, *r0*, *phi0*, *lam*, *Dphi*, *l_max=10*)

Return the gravitational wave emitted by a matter blob orbiting a Kerr black hole (semi-analytic computation based on a toy model surface density).

The surface density of the matter blob is that given by `surface_density_toy_model()`.

The gravitational wave is computed according to the formula

$$h = \frac{2\mu}{r} \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \frac{Z_{\ell m}^{\infty}(r_0)}{(m\omega_0)^2} \operatorname{sinc}\left(\frac{m}{2}\Delta\varphi\right) \operatorname{sinc}\left(\frac{3}{4}\varepsilon\, m\omega_0(1-a\omega_0)u\right) e^{-im(\omega_0 u+\phi_0)}\, {}_{-2}S_{\ell m}^{am\omega_0}(\theta,\varphi)$$

INPUT:

- `u` – retarded time coordinate of the observer (in units of M, the BH mass): $u = t - r_*$, where t is the Boyer-Lindquist time coordinate and r_* is the tortoise coordinate
- `theta` – Boyer-Lindquist colatitute θ of the observer
- `phi` – Boyer-Lindquist azimuthal coordinate ϕ of the observer
- `a` – BH angular momentum parameter (in units of M)
- `r0` – mean radius r_0 of the matter blob (Boyer-Lindquist coordinate)
- `phi0` – mean azimuthal angle ϕ_0 of the matter blob (Boyer-Lindquist coordinate)
- `lam` – radial extent λ of the matter blob
- `Dphi`– opening angle $\Delta\phi$ of the matter blob
- `l_max` – (default: 10) upper bound in the summation over the harmonic degree ℓ

OUTPUT:

- a pair `(hp, hc)`, where `hp` (resp. `hc`) is $(r/\mu)h_+$ (resp. $(r/\mu)h_\times$), μ being the blob's mass and r is the Boyer-Lindquist radial coordinate of the observer

EXAMPLES:

Schwarzschild black hole:

```
sage: from kerrgeodesic_gw import h_toy_model_semi_analytic
sage: a = 0
sage: r0, phi0, lam, Dphi = 6.5, 0, 0.6, 0.1
sage: u = 60.
sage: h_toy_model_semi_analytic(u, pi/4, 0., a, r0, phi0, lam, Dphi)  # tol 1.0e-13
(0.2999183296797872, 0.36916647790743246)
sage: hp, hc = _
```

Comparison with the exact value:

```
sage: from kerrgeodesic_gw import (h_blob, blob_mass,
....:                              surface_density_toy_model)
sage: param_surf_dens = [r0, phi0, lam, Dphi]
sage: integ_range = [6.2, 6.8, -0.05, 0.05]
sage: mu = blob_mass(a, surface_density_toy_model, param_surf_dens,
....:                integ_range)[0]
sage: hp0 = h_blob(u, pi/4, 0., a, surface_density_toy_model,
....:              param_surf_dens, integ_range)[0] / mu
sage: hc0 = h_blob(u, pi/4, 0., a, surface_density_toy_model,
....:              param_surf_dens, integ_range, mode='x')[0] / mu
sage: hp0, hc0  # tol 1.0e-13
(0.2951163078053617, 0.3743683023327848)
sage: (hp - hp0) / hp0  # tol 1.0e-13
0.01627162494047128
sage: (hc - hc0) / hc0  # tol 1.0e-13
-0.013894938201066784
```

`kerrgeodesic_gw.gw_blob.`**`surface_density_gaussian`**(*r*, *phi*, *param*)

Surface density of a matter blob with a Gaussian profile

INPUT:

- `r` – Boyer-Lindquist radial coordinate $\bar{r}$ in the matter blob
- `phi` – Boyer-Lindquist azimuthal coordinate $\bar{\phi}$ in the matter blob
- `param` – list of parameters defining the position and width of matter blob:
  - `param[0]`: mean radius r_0 (Boyer-Lindquist coordinate)
  - `param[1]`: mean azimuthal angle ϕ_0 (Boyer-Lindquist coordinate)
  - `param[2]`: width λ of the Gaussian profile
  - `param[3]` (optional): amplitude Σ_0; if not provided, then $\Sigma_0 = 1$ is used

OUTPUT:

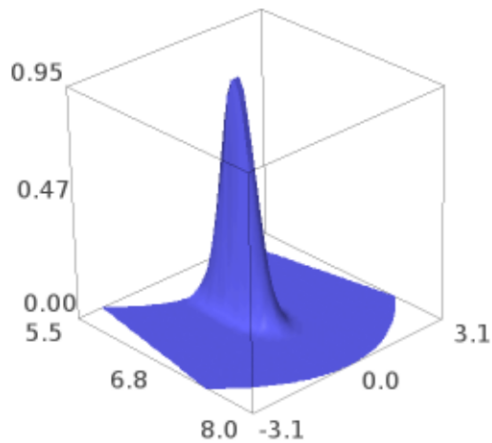
- surface density $\Sigma(\bar{r}, \bar{\phi})$

EXAMPLES:

```
sage: from kerrgeodesic_gw import surface_density_gaussian
sage: param = [6.5, 0., 0.3]
sage: surface_density_gaussian(6.5, 0, param)
1.0
sage: surface_density_gaussian(8., 0, param)  # tol 1.0e-13
1.3887943864964021e-11
sage: surface_density_gaussian(6.5, pi/16, param)  # tol 1.0e-13
1.4901161193847656e-08
```

3D representation: $z = \Sigma(\bar{r}, \bar{\phi})$ in terms of $x := \bar{r}\cos\bar{\phi}$ and $y := \bar{r}\sin\bar{\phi}$:

```
sage: s_plot = lambda r, phi: surface_density_gaussian(r, phi, param)
sage: r, phi, z = var('r phi z')
sage: plot3d(s_plot, (r, 6, 8), (phi, -0.4, 0.4),
....:        transformation=(r*cos(phi), r*sin(phi), z))
Graphics3d Object
```



Use with a non-default amplitude ($\Sigma_0 = 10^{-5}$):

```
sage: sigma0 = 1.e-5
sage: param = [6.5, 0., 0.3, sigma0]
sage: surface_density_gaussian(6.5, 0, param)
1e-05
```

kerrgeodesic_gw.gw_blob.**surface_density_toy_model**(*r*, *phi*, *param*)

Surface density of the toy model matter blob.

INPUT:

- `r` – Boyer-Lindquist radial coordinate $\bar{r}$ in the matter blob
- `phi` – Boyer-Lindquist azimuthal coordinate $\bar{\phi}$ in the matter blob
- `param` – list of parameters defining the position and extent of the matter blob:
  - `param[0]`: mean radius r_0 (Boyer-Lindquist coordinate)
  - `param[1]`: mean azimuthal angle ϕ_0 (Boyer-Lindquist coordinate)

- `param[2]`: radial extent λ
- `param[3]`: opening angle $\Delta\phi$
- `param[4]` (optional): amplitude Σ_0; if not provided, then $\Sigma_0 = 1$ is used

OUTPUT:

- surface density $\Sigma(\bar{r}, \bar{\phi})$

EXAMPLES:

Use with the default amplitude ($\Sigma_0 = 1$):

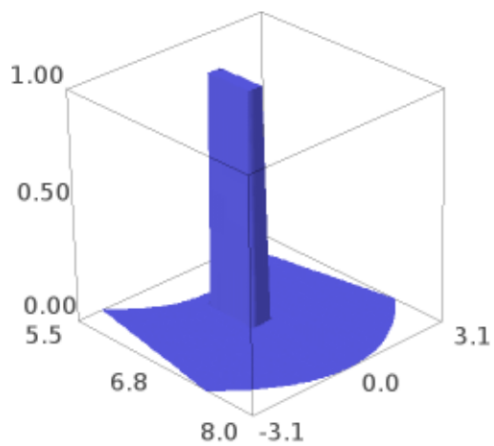
```
sage: from kerrgeodesic_gw import surface_density_toy_model
sage: param = [6.5, 0, 0.6, 0.1]
sage: surface_density_toy_model(6.5, 0, param)
1.0
sage: surface_density_toy_model(6.7, -0.04, param)
1.0
sage: surface_density_toy_model(6.3, 0.04, param)
1.0
sage: surface_density_toy_model(7, -0.06, param)
0.0
sage: surface_density_toy_model(5., 0.06, param)
0.0
```

3D representation: $z = \Sigma(\bar{r}, \bar{\phi})$ in terms of $x := \bar{r}\cos\bar{\phi}$ and $y := \bar{r}\sin\bar{\phi}$:

```
sage: s_plot = lambda r, phi: surface_density_toy_model(r, phi, param)
sage: r, phi, z = var('r phi z')
sage: plot3d(s_plot, (r, 6, 8), (phi, -0.4, 0.4),
....:        transformation=(r*cos(phi), r*sin(phi), z))
Graphics3d Object
```

Use with a non-default amplitude ($\Sigma_0 = 2$):

```
sage: sigma0 = 2.
sage: param = [6.5, 0, 0.6, 0.1, sigma0]
sage: surface_density_toy_model(6.5, 0, param)
2.0
```


1.00
0.50
0.00
5.5
6.8
8.0
-3.1
0.0
3.1

CHAPTER

SEVEN

LISA DETECTOR

Functions relative to the LISA detector.

kerrgeodesic_gw.lisa_detector.**power_spectral_density**(*freq*)

Return the effective power spectral density (PSD) of the detector noise at a given frequency.

INPUT:

- freq – frequency f (in Hz)

OUTPUT:

- effective power spectral density $S(f)$ (in Hz^{-1})

EXAMPLES:

```
sage: from kerrgeodesic_gw import lisa_detector
sage: Sn = lisa_detector.power_spectral_density
sage: Sn(1.e-1)  # tol 1.0e-13
3.3944027493062926e-39
sage: Sn(1.e-2)  # tol 1.0e-13
2.738383947022306e-40
sage: Sn(1.e-3)  # tol 1.0e-13
3.269807574220045e-38
```

kerrgeodesic_gw.lisa_detector.**power_spectral_density_RCLfit**(*freq*)

Return the effective power spectral density (PSD) of the detector noise at a given frequency, according to the analytical fit by Robson, Cornish and Liu, Arxiv 1803.01944

INPUT:

- freq – frequency f (in Hz)

OUTPUT:

- effective power spectral density $S(f)$ (in Hz^{-1})

EXAMPLES:

```
sage: from kerrgeodesic_gw import lisa_detector
sage: Sn = lisa_detector.power_spectral_density_RCLfit
sage: Sn(1.e-1)  # tol 1.0e-13
2.12858262120861e-39
sage: Sn(1.e-2)  # tol 1.0e-13
1.44307343517977e-40
sage: Sn(1.e-3)  # tol 1.0e-13
1.63410027259543e-38
```

kerrgeodesic_gw.lisa_detector.**strain_sensitivity**(*freq*)

Return LISA strain spectral sensitivity at a given frequency.

The strain spectral sensitivity is the square root of the effective noise power spectral density (cf. *power_spectral_density()*).

INPUT:

- `freq` – frequency f (in Hz)

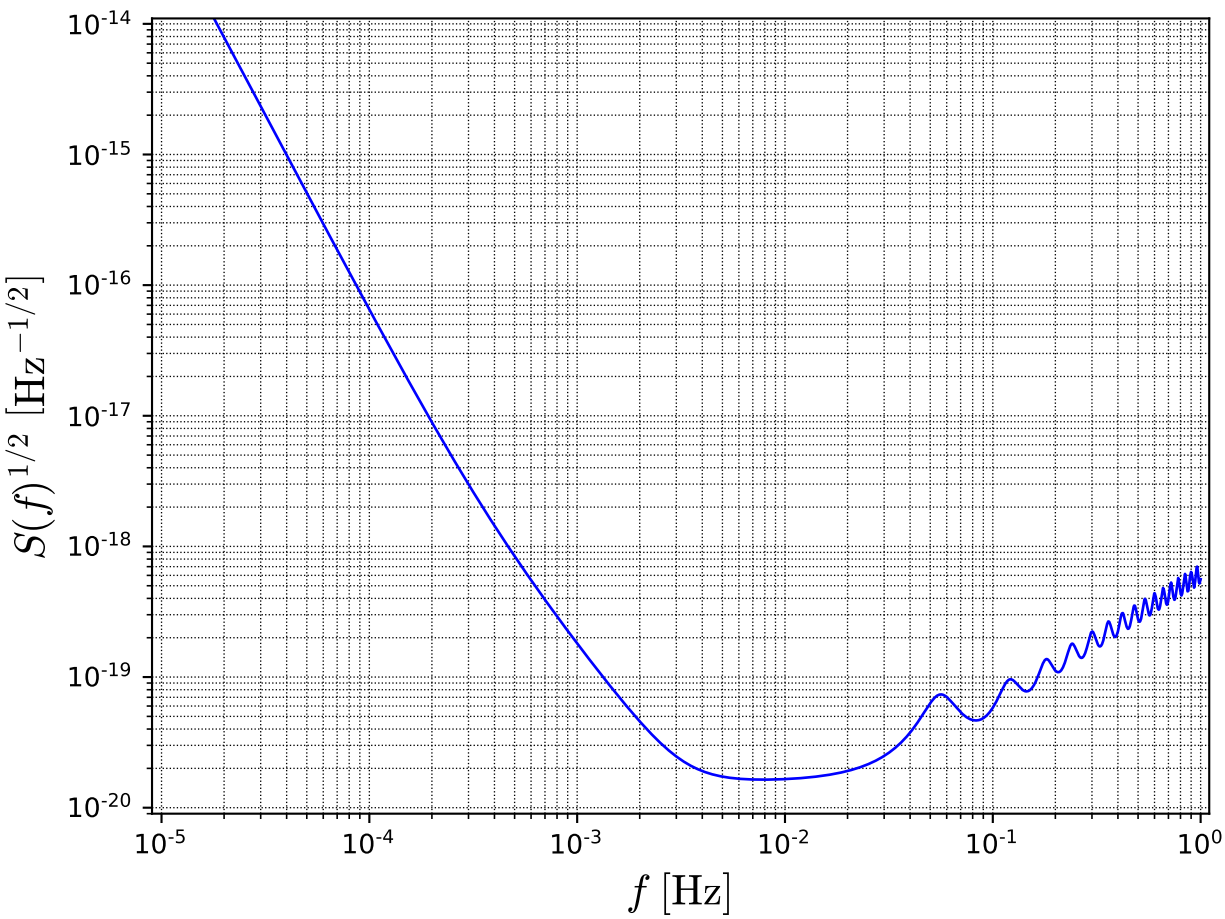
OUTPUT:

- strain sensitivity $S(f)^{1/2}$ (in $\mathrm{Hz}^{-1/2}$)

EXAMPLES:

```
sage: from kerrgeodesic_gw import lisa_detector
sage: hn = lisa_detector.strain_sensitivity
sage: hn(1.e-1)  # tol 1.0e-13
5.82615031500758e-20
sage: hn(1.e-2)  # tol 1.0e-13
1.654806317072275e-20
sage: hn(1.e-3)  # tol 1.0e-13
1.8082609253700212e-19
```

```
sage: plot_loglog(hn, (1e-5, 1), plot_points=2000, ymin=1e-20, ymax=1e-14,
....:             axes_labels=[r"$f\ [\mathrm{Hz}]$",
....:                          r"$S(f)^{1/2} \ \left[\mathrm{Hz}^{-1/2}\right]$"],
....:             gridlines='minor', frame=True, axes=False)
Graphics object consisting of 1 graphics primitive
```



CHAPTER

EIGHT

SIGNAL PROCESSING

Signal processing

`kerrgeodesic_gw.signal_processing.`**`fourier`**(*signal*)

Compute the Fourier transform of a time signal.

The *Fourier transform* of the time signal $h(t)$ is defined by

$$\tilde{h}(f) = \int_{-\infty}^{+\infty} h(t)\, \mathrm{e}^{-2\pi \mathrm{i} f t}\, \mathrm{d}t$$

INPUT:

- `signal` – list of pairs $(t, h(t))$, where t is the time and $h(t)$ is the signal at t. *NB*: the sampling in t must be uniform

OUTPUT:

- a numerical approximation (FFT) of the Fourier transform, as a list of pairs $(f, \tilde{h}(f))$

EXAMPLES:

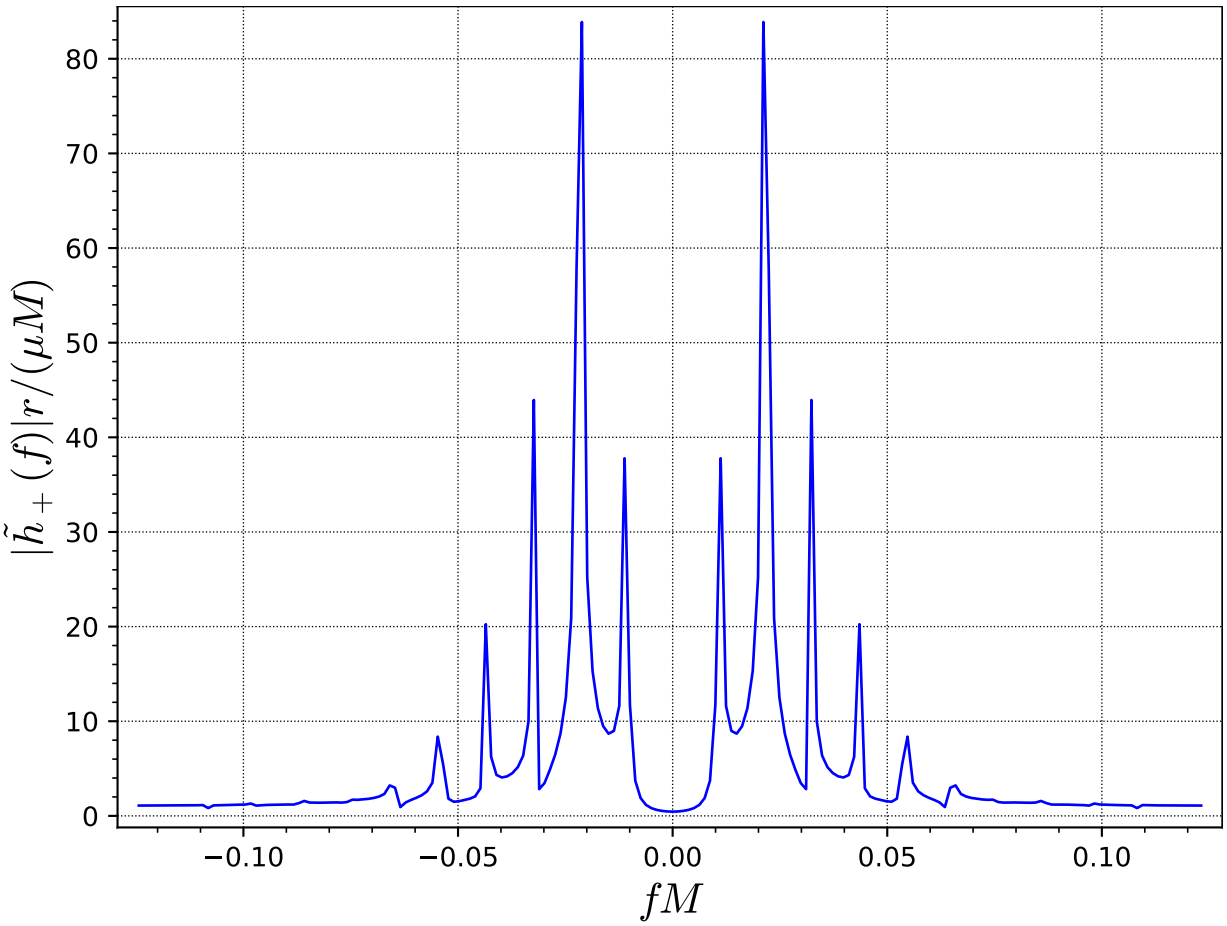
Fourier transform of the h_+ signal from a particle orbiting at the ISCO of a Schwarzschild black hole:

```
sage: from kerrgeodesic_gw import h_particle_signal, fourier
sage: a, r0 = 0., 6.
sage: theta, phi = pi/2, 0
sage: sig = h_particle_signal(a, r0, theta, phi, 0., 800., nb_points=200)
sage: sig[:5]  # tol 1.0e-13
[(0.000000000000000, 0.1536656546005028),
 (4.02010050251256, 0.03882960191740132),
 (8.04020100502512, -0.0791773803745108),
 (12.0603015075377, -0.18368354016995483),
 (16.0804020100502, -0.25796165580196795)]
sage: ft = fourier(sig)
sage: ft[:5]   # tol 1.0e-13
[(-0.12437500000000001, (1.0901773159466113+0j)),
 (-0.12313125000000001, (1.0901679813096925+0.021913447210468399j)),
 (-0.12188750000000001, (1.0901408521337845+0.043865034665617614j)),
 (-0.12064375000000001, (1.0900987733011678+0.065894415355341171j)),
 (-0.11940000000000001, (1.0900473072498438+0.088044567109304195j))]
```

Plot of the norm of the Fourier transform:

```
sage: ft_norm = [(f, abs(hf)) for (f, hf) in ft]
sage: line(ft_norm, axes_labels=[r'$f M$', r'$|\tilde{h}_+(f)| r/(\mu M)$'],
....:      gridlines=True, frame=True, axes=False)
Graphics object consisting of 1 graphics primitive
```

The first peak is at the orbital frequency of the particle:


80
70
60
50
40
30
20
10
0
$|\tilde{h}_+(f)|r/(\mu M)$
−0.10
−0.05
0.00
0.05
0.10
fM

```
sage: f0 = n(1/(2*pi*r0^(3/2))); f0
0.0108291222393566
```

while the highest peak is at twice this frequency ($m = 2$ mode).

kerrgeodesic_gw.signal_processing.**max_detectable_radius**(*a*, *mu*, *theta*, *psd*, *BH_time_scale*, *distance*, *t_obs_yr=1*, *snr_threshold=10*, *r_min=None*, *r_max=200*, *m_min=1*, *m_max=None*, *approximation=None*)

Compute the maximum orbital radius $r_{0,\max}$ at which a particle of given mass is detectable.

INPUT:

- `a` – BH angular momentum parameter (in units of M, the BH mass)
- `mu` – mass of the orbiting object, in solar masses
- `theta` – Boyer-Lindquist colatitute θ of the observer
- `psd` – function with a single argument (f) representing the detector's one-sided noise power spectral density $S_n(f)$ (see e.g. `lisa_detector.power_spectral_density()`)
- `BH_time_scale` – value of M in the same time unit as $S_n(f)$; if $S_n(f)$ is provided in Hz^{-1}, then `BH_time_scale` must be M expressed in seconds.
- `distance` – distance r to the detector, in parsecs
- `t_obs_yr` – (default: 1) observation period, in years
- `snr_threshold` – (default: 10) signal-to-noise value above which a detection is claimed
- `r_min` – (default: `None`) lower bound of the search interval for $r_{0,\max}$ (in units of M); if `None` then the ISCO value is used
- `r_max` – (default: 200) upper bound of the search interval for $r_{0,\max}$ (in units of M)
- `m_min` – (default: 1) lower bound in the summation over the Fourier mode m
- `m_max` – (default: `None`) upper bound in the summation over the Fourier mode m; if `None`, `m_max` is set to 10 for $r_0 \leq 20M$ and to 5 for $r_0 > 20M$
- `approximation` – (default: `None`) string describing the computational method for the signal; allowed values are
  - `None`: exact computation
  - `'quadrupole'`: quadrupole approximation; see `gw_particle.h_particle_quadrupole()`
  - `'1.5PN'` (only for `a=0`): 1.5-post-Newtonian expansion following E. Poisson, Phys. Rev. D **47**, 1497 (1993) [doi:10.1103/PhysRevD.47.1497]

OUTPUT:

- Boyer-Lindquist radius (in units of M) of the most remote orbit for which the signal-to-noise ratio is larger than `snr_threshold` during the observation time `t_obs_yr`

EXAMPLES:

Maximum orbital radius for LISA detection of a 1 solar-mass object orbiting Sgr A* observed by LISA, assuming a BH spin $a = 0.9M$ and a vanishing inclination angle:

```
sage: from kerrgeodesic_gw import (max_detectable_radius, lisa_detector,
....:                              astro_data)
sage: a = 0.9
sage: mu = 1
```

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```
sage: theta = 0
sage: psd = lisa_detector.power_spectral_density_RCLfit
sage: BH_time_scale = astro_data.SgrA_mass_s
sage: distance = astro_data.SgrA_distance_pc
sage: max_detectable_radius(a, mu, theta, psd, BH_time_scale, distance)  # tol 1.0e-13
46.98292057022975
```

Lowering the SNR threshold to 5:

```
sage: max_detectable_radius(a, mu, theta, psd, BH_time_scale, distance,  # tol 1.0e-13
....:                       snr_threshold=5)
53.503386416644645
```

Lowering the data acquisition time to 1 day:

```
sage: max_detectable_radius(a, mu, theta, psd, BH_time_scale, distance,  # tol 1.0e-13
....:                       t_obs_yr=1./365.25)
27.158725683511463
```

Assuming an inclination angle of $\pi/2$:

```
sage: theta = pi/2
sage: max_detectable_radius(a, mu, theta, psd, BH_time_scale, distance)  # tol 1.0e-13
39.81826500968452
```

Using the 1.5-PN approximation (`a` has to be zero and `m_max` has to be at most 5):

```
sage: a = 0
sage: max_detectable_radius(a, mu, theta, psd, BH_time_scale,  # tol 1.0e-13
....:                       distance, approximation='1.5PN', m_max=5)
39.74273792957649
```

`kerrgeodesic_gw.signal_processing.`**`read_signal`**(*filename*, *dirname=None*)

Read a signal from a data file.

INPUT:

- `filename` – string; name of the file
- `dirname` – (default: None) string; name of directory where the file is located

OUTPUT:

- signal as a list of pairs $(t, h(t))$

EXAMPLES:

```
sage: from kerrgeodesic_gw import save_signal, read_signal
sage: sig0 = [(RDF(i/10), RDF(sin(i/5))) for i in range(5)]
sage: from sage.misc.temporary_file import tmp_filename
sage: filename = tmp_filename(ext='.dat')
sage: save_signal(sig0, filename)
sage: sig = read_signal(filename)
sage: sig   # tol 1.0e-13
[(0.0, 0.0),
 (0.1, 0.19866933079506122),
 (0.2, 0.3894183423086505),
 (0.3, 0.5646424733950354),
 (0.4, 0.7173560908995228)]
```

A test:

```
sage: sig == sig0
True
```

kerrgeodesic_gw.signal_processing.**save_signal**(*sig*, *filename*, *dirname=None*)

Write a signal in a data file.

INPUT:

- `sig` – signal as a list of pairs $(t, h(t))$
- `filename` – string; name of the file
- `dirname` – (default: None) string; name of directory where the file is located

EXAMPLES:

```
sage: from kerrgeodesic_gw import save_signal, read_signal
sage: sig = [(RDF(i/10), RDF(sin(i/5))) for i in range(5)]
sage: sig   # tol 1.0e-13
[(0.0, 0.0),
 (0.1, 0.19866933079506122),
 (0.2, 0.3894183423086505),
 (0.3, 0.5646424733950354),
 (0.4, 0.7173560908995228)]
sage: from sage.misc.temporary_file import tmp_filename
sage: filename = tmp_filename(ext='.dat')
sage: save_signal(sig, filename)
```

A test:

```
sage: sig1 = read_signal(filename)
sage: sig1 == sig
True
```

kerrgeodesic_gw.signal_processing.**signal_to_noise**(*signal*, *time_scale*, *psd*, *fmin*, *fmax*, *scale=1*, *interpolation='linear'*, *quad_epsrel=1e-05*, *quad_limit=500*)

Evaluate the signal-to-noise ratio of a signal observed in a detector of a given power spectral density.

The *signal-to-noise ratio* ρ of the time signal $h(t)$ is computed according to the formula

$$\rho^2 = 4 \int_0^{+\infty} \frac{|\tilde{h}(f)|^2}{S_n(f)} \, \mathrm{d}f \tag{8.1}$$

where $\tilde{h}(f)$ is the Fourier transform of $h(t)$ (see *fourier()*) and $S_n(f)$ is the detector's one-sided noise power spectral density (see e.g. *lisa_detector.power_spectral_density()*).

INPUT:

- `signal` – list of pairs $(t, h(t))$, where t is the time and $h(t)$ is the signal at t. *NB*: the sampling in t must be uniform
- `time_scale` – value of t unit in terms of $S_n(f)$ unit; if $S_n(f)$ is provided in Hz^{-1}, then `time_scale` must be the unit of t in `signal` expressed in seconds.
- `psd` – function with a single argument (f) representing the detector's one-sided noise power spectral density $S_n(f)$
- `fmin` – lower bound used instead of 0 in the integral (8.1)
- `fmax` – upper bound used instead of $+\infty$ in the integral (8.1)
- `scale` – (default: 1) scale factor by which $h(t)$ must be multiplied to get the actual signal

- `interpolation` – (default: `'linear'`) string describing the type of interpolation used to evaluate $|h(f)|^2$ from the list resulting from the FFT of `signal`; allowed values are
  - `'linear'`: linear interpolation between two data points
  - `'spline'`: cubic spline interpolation
- `quad_epsrel` – (default: `1.e-6`) relative error tolerance in the computation of the integral (8.1)
- `quad_limit` – (default: `500`) upper bound on the number of subintervals used in the adaptive algorithm to compute the integral (this corresponds to the argument `limit` of SciPy's function quad)

OUTPUT:

- the signal-to-noise ratio ρ computed via the integral (8.1), with the boundaries 0 and $+\infty$ replaced by respectively `fmin` and `fmax`.

EXAMPLES:

Let us evaluate the SNR of the gravitational signal generated by a 1-solar mass object orbiting at the ISCO of Sgr A* observed by LISA during 1 day. We need the following functions:

```
sage: from kerrgeodesic_gw import (h_particle_signal, signal_to_noise,
....:                              lisa_detector, astro_data)
```

We model Sgr A* as a Schwarzschild black hole and we consider that the signal is constituted by the mode $h_+(t)$ observed in the orbital plane:

```
sage: a, r0 = 0., 6.
sage: theta, phi = pi/2, 0
sage: tmax = 24*3600/astro_data.MSgrA_s  # 1 day in units of Sgr A* mass
sage: h = h_particle_signal(a, r0, theta, phi, 0., tmax, mode='+',
....:                       nb_points=4000)
```

The signal-to-noise ratio is then computed as:

```
sage: time_scale = astro_data.MSgrA_s  # Sgr A* mass in seconds
sage: psd = lisa_detector.power_spectral_density_RCLfit
sage: fmin, fmax = 1e-5, 5e-3
sage: mu_ov_r = astro_data.Msol_m / astro_data.dSgrA  # mu/r
sage: signal_to_noise(h, time_scale, psd, fmin, fmax,     # tol 1.0e-13
....:                 interpolation='spline', scale=mu_ov_r)
7583.104124621172
```

Signal-to-noise for a signal computed at the quadrupole approximation:

```
sage: h = h_particle_signal(a, r0, theta, phi, 0., tmax, mode='+',
....:                       nb_points=4000, approximation='quadrupole')
sage: signal_to_noise(h, time_scale, psd, fmin, fmax,     # tol 1.0e-13
....:                 interpolation='spline', scale=mu_ov_r)
5383.264004811493
```

`kerrgeodesic_gw.signal_processing.`**`signal_to_noise_particle`**(*a*, *r0*, *theta*, *psd*, *t_obs*, *BH_time_scale*, *m_min=1*, *m_max=None*, *scale=1*, *approximation=None*)

Evaluate the signal-to-noise ratio of gravitational radiation emitted by a single orbiting particle observed in a detector of a given power spectral density.

INPUT:

- `a` – BH angular momentum parameter (in units of M, the BH mass)
- `r0` – Boyer-Lindquist radius of the orbit (in units of M)
- `theta` – Boyer-Lindquist colatitute θ of the observer

- `psd` – function with a single argument (f) representing the detector's one-sided noise power spectral density $S_n(f)$ (see e.g. *lisa_detector.power_spectral_density()*)
- `t_obs` – observation period, in the same time unit as $S_n(f)$
- `BH_time_scale` – value of M in the same time unit as $S_n(f)$; if $S_n(f)$ is provided in Hz^{-1}, then `BH_time_scale` must be M expressed in seconds.
- `m_min` – (default: 1) lower bound in the summation over the Fourier mode m
- `m_max` – (default: `None`) upper bound in the summation over the Fourier mode m; if `None`, `m_max` is set to 10 for $r_0 \leq 20M$ and to 5 for $r_0 > 20M$
- `scale` – (default: 1) scale factor by which $h(t)$ must be multiplied to get the actual signal; this should by μ/r, where μ is the particle mass and r the radial coordinate of the detector
- `approximation` – (default: `None`) string describing the computational method for the signal; allowed values are
  - `None`: exact computation
  - `'quadrupole'`: quadrupole approximation; see *gw_particle.h_particle_quadrupole()*
  - `'1.5PN'` (only for `a=0`): 1.5-post-Newtonian expansion following E. Poisson, Phys. Rev. D **47**, 1497 (1993) [doi:10.1103/PhysRevD.47.1497]

OUTPUT:

- the signal-to-noise ratio ρ

EXAMPLES:

Let us evaluate the SNR of the gravitational signal generated by a 1-solar mass object orbiting at the ISCO of Sgr A* observed by LISA during 1 day:

```
sage: from kerrgeodesic_gw import (signal_to_noise_particle,
....:                              lisa_detector, astro_data)
sage: a, r0 = 0., 6.
sage: theta = pi/2
sage: t_obs = 24*3600  # 1 day in seconds
sage: BH_time_scale = astro_data.SgrA_mass_s  # Sgr A* mass in seconds
sage: psd = lisa_detector.power_spectral_density_RCLfit
sage: mu_ov_r = astro_data.Msol_m / astro_data.dSgrA  # mu/r
sage: signal_to_noise_particle(a, r0, theta, psd, t_obs,  # tol 1.0e-13
....:                          BH_time_scale, scale=mu_ov_r)
7565.450402023329
```

Using the quadrupole approximation:

```
sage: signal_to_noise_particle(a, r0, theta, psd, t_obs,  # tol 1.0e-13
....:                          BH_time_scale, scale=mu_ov_r,
....:                          approximation='quadrupole')
5230.216539094391
```

Using the 1.5-PN approximation (`m_max` has to be at most 5):

```
sage: signal_to_noise_particle(a, r0, theta, psd, t_obs,  # tol 1.0e-13
....:                          BH_time_scale, scale=mu_ov_r,
....:                          approximation='1.5PN', m_max=5)
7601.135299165022
```

For large values of r_0, the 1.5-PN approximation and the quadrupole one converge:

```
sage: r0 = 100
sage: signal_to_noise_particle(a, r0, theta, psd, t_obs,  # tol 1.0e-13
....:                          BH_time_scale, scale=mu_ov_r,
....:                          approximation='quadrupole')
0.003053021617751869
sage: signal_to_noise_particle(a, r0, theta, psd, t_obs,  # tol 1.0e-13
....:                          BH_time_scale, scale=mu_ov_r,
....:                          approximation='1.5PN')
0.003144006483338098
```

```
sage: r0 = 1000
sage: signal_to_noise_particle(a, r0, theta, psd, t_obs,  # tol 1.0e-13
....:                          BH_time_scale, scale=mu_ov_r,
....:                          approximation='quadrupole')
9.663153184393872e-09
sage: signal_to_noise_particle(a, r0, theta, psd, t_obs,  # tol 1.0e-13
....:                          BH_time_scale, scale=mu_ov_r,
....:                          approximation='1.5PN')
9.686830661779979e-09
```

CHAPTER

NINE

ASTRONOMICAL DATA

Astronomical data.

Having imported the `astro_data` module, type

- `astro_data.<TAB>`, where `<TAB>` stands for the tabulation key, to get the list of available data
- `astro_data.??` to see the sources of the numerical values

EXAMPLES:

Fundamental constants (in SI units):

```
sage: from kerrgeodesic_gw import astro_data
sage: astro_data.G
6.6743e-11
sage: astro_data.c
299792458.0
```

Solar mass in kg:

```
sage: astro_data.solar_mass_kg
1.98848e+30
```

Solar mass in meters (geometrized units):

```
sage: astro_data.solar_mass_m  # tol 1.0e-13
1476.6771171937082
```

Solar mass in seconds (geometrized units):

```
sage: astro_data.solar_mass_s  # tol 1.0e-13
4.925664664965348e-06
```

Mass of Sgr A* in solar masses:

```
sage: astro_data.SgrA_mass_sol
4100000.0
```

Mass of Sgr A* in kg:

```
sage: astro_data.SgrA_mass_kg  # tol 1.0e-13
8.152768e+36
```

Mass of Sgr A* in meters (geometrized units):

```
sage: astro_data.SgrA_mass_m  # tol 1.0e-13
6054376180.4942045
```

Mass of Sgr A* in seconds (geometrized units):

```
sage: astro_data.SgrA_mass_s  # tol 1.0e-13
20.19522512635793
```

Distance to Sgr A* in parsecs and meters, respectively:

```
sage: astro_data.SgrA_distance_pc
8120.0
sage: astro_data.SgrA_distance_m  # tol 1.0e-13
2.5055701961709902e+20
```

CHAPTER

TEN

UTILITIES

Utilities

This module contains some utility functions. For the moment, there is only

- *graphics_inset()*: generates plots with insets

kerrgeodesic_gw.utilities.**graphics_inset**(*main*, *inset*, *position*, ***kwds*)

Insert a graphics object as an inset into another graphics object.

This function compensates for the (current) lack of the inset functionality in SageMath graphics.

The output is not a SageMath graphics object but a Matplotlib object, of type `Figure`.

INPUT:

- `main` – a SageMath graphics object
- `inset` – a SageMath graphics object to be inserted into `main`
- `position` – a 4-uple `(left, bottom, width, height)` specifying the position and size of the inset (all quantities are in fractions of figure width and height)
- `kwds` – options passed to method `matplotlib.figure.Figure.add_axes`

OUTPUT:

- a `matplotlib.figure.Figure` object

EXAMPLES:

```
sage: from kerrgeodesic_gw.utilities import graphics_inset
sage: g1 = plot(x*sin(1/x), (x, -2, 2), axes_labels=[r"$x$", r"$y$"])
sage: g2 = plot(x*sin(1/x), (x, -0.05, 0.05),
....:          axes_labels=[r"$x$", r"$y$"],fontsize=8, frame=True)
sage: figure = graphics_inset(g1, g2, (0.72, 0.4, 0.25, 0.25))
sage: figure
<Figure size 640x480 with 2 Axes>
```

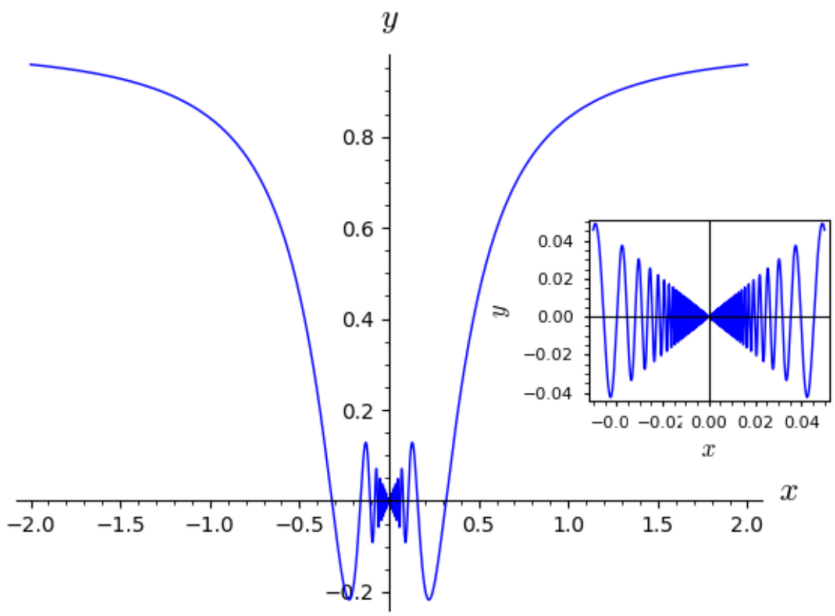
The output is a Matplotlib object, of type `Figure`:

```
sage: type(figure)
<class 'matplotlib.figure.Figure'>
```

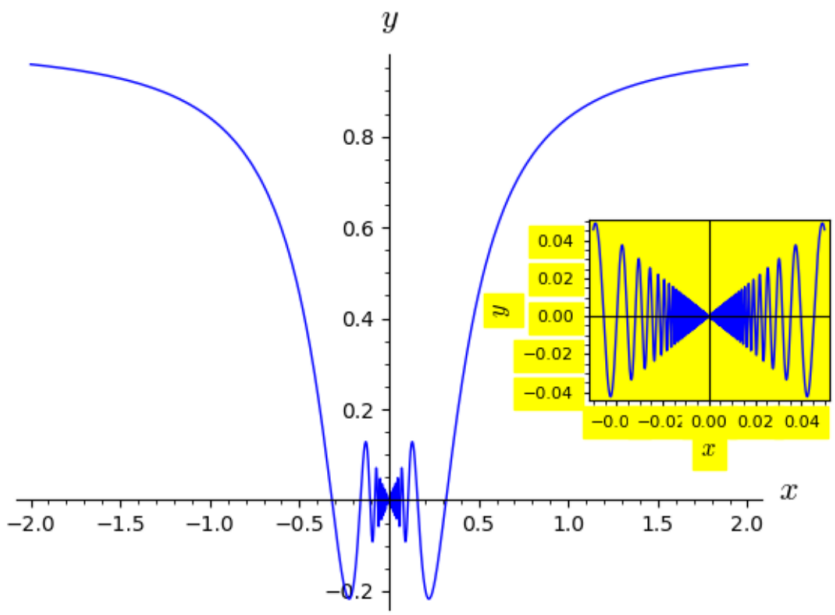
It can be saved into a pdf file via the method `savefig`:

```
sage: figure.savefig("figure.pdf")
```

An example with a Matplotlib keyword:


y
0.8
0.6
0.4
0.2
−0.2
−2.0
−1.5
−1.0
−0.5
0.5
1.0
1.5
2.0
x
0.04
0.02
0.00
−0.02
−0.04
−0.0
−0.02
0.00
0.02
0.04
y
x

```
sage: figure = graphics_inset(g1, g2, (0.72, 0.4, 0.25, 0.25),
....:                         facecolor='yellow')
sage: figure
<Figure size 640x480 with 2 Axes>
```



PYTHON MODULE INDEX

k

INDEX

I

K

M

O

P

R

S

Z